Effective Bounds on Singular Surfaces in Positive Characteristic

J A K U B   W I T A S Z E K

ABSTRACT. Using the theory of Frobenius singularities, we show that $13mK_X + 45mA$ is very ample for an ample Cartier divisor A on a Kawamata log terminal surface X with Gorenstein index m, defined over an algebraically closed field of characteristic $p > 5$.

1. Introduction

The positivity of line bundles is a fundamental topic of research in algebraic geometry. Showing the base point freeness or very ampleness of line bundles allows for the description of the geometry of algebraic varieties.

The motivation for this paper centers around two questions. The first one is the following: given an ample Cartier divisor A, find an effective $n \in \mathbb{N}$ for which nA is very ample. A famous theorem of Matsusaka states that we can find such $n \in \mathbb{N}$ that depends only on the Hilbert polynomial of A when the variety is smooth and the characteristic of the field is equal to zero [20]. This theorem plays a fundamental role in constructing moduli spaces of polarized varieties. In positive characteristic, Kollár proved the same statement for normal surfaces [11, Thm. 2.1.2].

The second question motivating the results of this paper is the famous Fujita conjecture, which, in characteristic zero, is proved only for curves and surfaces.

CONJECTURE 1.1 (Fujita conjecture). *Let X be a smooth projective variety of dimension n, and let A be an ample Cartier divisor on X. Then $K_X + (n + 2)A$ is very ample.*

Fujita-type results play a vital role in understanding the geometry of algebraic varieties.

In positive characteristic, the conjecture is known only for curves and for surfaces that are neither of general type nor quasi-elliptic. This follows from a result of Shepherd–Barron, which says that on such surfaces, rank two vector bundles that do not satisfy Bogomolov inequality are unstable [28, Thm. 7]. Indeed, the celebrated proof by Reider of the Fujita conjecture for characteristic zero surfaces can be, in such a case, applied without any modifications (see [35; 21]).

Given lack of any progress for positive characteristic surfaces of general type, Di Cerbo and Fanelli [4] undertook a different approach to the problem. They proved among other things that $2K_X + 4A$ is very ample if A is ample and X is a smooth surface of general type in characteristic $p \geq 3$.