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Curve Arrangements, Pencils, and Jacobian Syzygies

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Abstract. Let $\mathcal{C} : f = 0$ be a curve arrangement in the complex projective plane. If $\mathcal{C}$ contains a curve subarrangement consisting of at least three members in a pencil, then we obtain an explicit syzygy among the partial derivatives of the homogeneous polynomial f. In many cases, this observation reduces the question about the freeness or the nearly freeness of $\mathcal{C}$ to an easy computation of Tjurina numbers. We also discuss some consequences for Terao's conjecture in the case of line arrangements and the asphericity of some complements of geometrically constructed free curves.

1. Introduction

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in the variables x, y, z with complex coefficients, and let $\mathcal{C} : f = 0$ be a reduced curve of degree d in the complex projective plane $\mathbb{P}^2$. The minimal degree of a Jacobian syzygy for f is the integer $\mathrm{mdr}(f)$ defined to be the smallest integer $r \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0 \tag{1.1}$$

among the partial derivatives f_x, f_y, and f_z of f with coefficients a, b, c in S_r, the vector space of homogeneous polynomials of degree r. The knowledge of the invariant $\mathrm{mdr}(f)$ allows us to decide if the curve $\mathcal{C}$ is free or nearly free by a simple computation of the total Tjurina number $\tau(\mathcal{C})$; see [9; 5], and Theorems 1.12 and 1.14 for nice geometric applications. Recall that a curve $\mathcal{C}$ as before is free (resp. nearly free) if and only if $\tau(\mathcal{C}) = (d-1)^2 - r(d-r-1)$ (resp. $\tau(\mathcal{C}) = (d-1)^2 - r(d-r-1) - 1$), where $r = \mathrm{mdr}(f)$. These conditions tell that the minimal resolution of the graded S-module of Jacobian syzygies $AR(f) \subset S^3$ consisting of all relations of type (1.1) satisfies certain properties; see [5] for details.

When $\mathcal{C}$ is a free (resp. nearly free) curve in the complex projective plane $\mathbb{P}^2$ such that $\mathcal{C}$ is not a union of lines passing through one point, then the exponents of $\mathcal{A}$ denoted by $d_1 \leq d_2$ satisfy $d_1 = \mathrm{mdr}(f) \geq 1$, and we have

$$d_1 + d_2 = d - 1 \tag{1.2}$$

(resp. $d_1 + d_2 = d$). Moreover, all the pairs d_1, d_2 satisfying these conditions may occur as exponents; see [8]. For more on free hypersurfaces and free hyperplane arrangements, see [18; 15; 24; 19]. A useful result is the following.

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