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Cohomology Support Loci of Local Systems

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Abstract. The support S of Sabbah's specialization complex is a simultaneous generalization of the set of eigenvalues of the monodromy on Deligne's nearby cycles complex, of the support of the Alexander modules of an algebraic knot, and of certain cohomology support loci. Moreover, it equals conjecturally the image under the exponential map of the zero locus of the Bernstein–Sato ideal. Sabbah showed that S is contained in a union of translated subtori of codimension one in a complex affine torus. Budur and Wang showed recently that S is a union of torsion-translated subtori. We show here that S is always a hypersurface and that it admits a formula in terms of log resolutions. As an application, we give a criterion in terms of log resolutions for the (semi)simplicity as perverse sheaves, or as regular holonomic D-modules, of the direct images of rank one local systems under an open embedding. For hyperplane arrangements, this criterion is combinatorial.

1. Introduction

1.1. Cohomology Support Loci

For a topological space T, let $M_B(T)$ be the moduli space of rank 1 $\mathbb{C}$-local systems on T. The *cohomology support loci of* T are defined as

$$\mathcal{V}(T) = \{L \in M_B(T) \mid \dim H^\bullet(T, L) \neq 0\}$$

and are homotopy invariants of T. It was shown recently in [BW15a; BW15b] that $\mathcal{V}(T)$ are finite unions of torsion-translated affine subtori of the affine algebraic group $M_B(T) \cong Hom(H_1(T,\mathbb{Z}), \mathbb{C}^*)$ if T is a smooth complex quasi-projective algebraic variety or a small ball complement of the germ of a complex analytic set in a complex manifold. It remains though a difficult task to compute cohomology support loci. This article is an application of the structure result for cohomology support loci.

Let $j : U \to X$ be the open embedding in a complex manifold X of the complement of a hypersurface $f^{-1}(0)$, where $f : X \to \mathbb{C}$ is an noninvertible analytic function. For $x \in f^{-1}(0)$, let U_x be the complement in a small ball in X centered at x of $f^{-1}(0)$. Then $M_B(U_x) \cong (\mathbb{C}^*)^r$, where r is the number of analytic branches of f at x. In this paper, we show that if we take the union, in a certain sense, of the cohomology support loci $\mathcal{V}(U_x)$ for all points $x \in f^{-1}(0)$, then the

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