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# Slopes of Fibered Surfaces with a Finite Cyclic Automorphism

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Abstract. We study slopes of finite cyclic covering fibrations of a fibered surface. We give the best possible lower bound of the slope of these fibrations. We also give the slope equality of finite cyclic covering fibrations of a ruled surface and observe the local concentration of the global signature of these surfaces on a finite number of fiber germs. We also give an upper bound of the slope of finite cyclic covering fibrations of a ruled surface.

## Introduction

Let $f: S \to B$ be a surjective morphism from a complex smooth projective surface $S$ to a smooth projective curve $B$ with connected fibers. The datum $(S, f, B)$ or simply $f$ is called a *fibered surface* or a fibration. A fibered surface $f$ is said to be *relatively minimal* if there exist no $(-1)$-curves contained in fibers of $f$, where a $(-1)$-curve is a nonsingular rational curve with self-intersection number $-1$. The genus $g$ of a fibered surface $f$ is defined to be that of a general fiber of $f$. We put $K_f = K_S - f^*K_B$ and call it the relative canonical bundle.

Assume that $f: S \to B$ is a relatively minimal fibration of genus $g \geq 2$ and consider the following three relative invariants:

$$\begin{aligned} \chi_f &:= \chi(\mathcal{O}_S) - (g-1)(b-1), \\ K_f^2 &= K_S^2 - 8(g-1)(b-1), \\ e_f &:= e(S) - 4(g-1)(b-1), \end{aligned}$$

where $b$ and $e(S)$ respectively denote the genus of the base curve $B$ and the topological Euler characteristic of $S$. Then the following are well known:

- (Noether) $12\chi_f = K_f^2 + e_f$.
- (Arakelov) $K_f$ is nef.
- (Ueno) $\chi_f \geq 0$, and $\chi_f = 0$ if and only if $f$ is locally trivial (i.e., a holomorphic fiber bundle).
- (Segre) $e_f \geq 0$, and $e_f = 0$ if and only if $f$ is smooth.

When $f$ is not locally trivial, we put

$$\lambda_f = \frac{K_f^2}{\chi_f}$$

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