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The Linear Strand of Determinantal Facet Ideals

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Abstract. Let X be an $(m \times n)$-matrix of indeterminates, and let J be the ideal generated by a set $\mathcal{S}$ of maximal minors of X. We construct the linear strand of the resolution of J. This linear strand is determined by the clique complex of the m-clutter corresponding to the set $\mathcal{S}$. As a consequence, we obtain explicit formulas for the graded Betti numbers $\beta_{i,i+m}(J)$ for all $i \geq 0$. We also determine all sets $\mathcal{S}$ for which J has a linear resolution.

Introduction

In this paper, we consider ideals generated by an arbitrary set of maximal minors of an $(m \times n)$-matrix of indeterminates. Such ideals have first been considered when $m = 2$, in which case the set of minors (which is a set of binomials) is in bijection with the edges of a graph G and therefore is called the binomial edge ideal of G. This class of ideals has first been considered in [13] and [21]. In [13], the relevance of such ideals for algebraic statistics has been stressed. In the sequel, binomial edge ideals have been studied in numerous papers with an attempt to better understand their algebraic and homological properties; see, for example, [8; 9; 15; 24] and [25]. In some particular cases, the resolution of such ideals has been determined, and upper bounds for their regularity have been given; see [4; 11; 14; 16; 18; 22] and [23].

When $m > 2$, these ideals are called determinantal facet ideals. Here their generators are in bijection with the facets of a pure simplicial complex of dimension $m - 1$. These ideals were introduced and first studied in [10]. Less is known about the resolution of determinantal facet ideals, even less than in the case of binomial edge ideals. Apart from a very special case considered in [19], the resolution of a determinantal facet ideal is only known when the underlying simplicial complex is a simplex, in which case the Eagon–Northcott complex provides a resolution.

One of the motivations for writing this paper was a conjecture made in [14] by the second and third authors of this paper. Given a finite simple graph G, let J_G be its binomial edge ideal. Then the graded Betti numbers $\beta_{i,i+2}(J_G)$ give the ranks of the corresponding free modules of the linear strand of J_G. The conjecture made in [14] says that $\beta_{i,i+2}(J_G) = (i+1) f_{i+1}(\Delta(G))$ for all $i \geq 0$. Here $\Delta(G)$ is the clique complex of G, and $f_{i+1}(\Delta(G))$ is the number of faces of $\Delta(G)$ of dimension $i + 1$. In this paper, we not only prove this conjecture but also prove an

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