Michigan Math. J. 66 (2017), 37–47

# Twisting of Composite Torus Knots

Mohamed Ait Nouh

Abstract. We prove that the family of connected sums of torus knots $T(2,p)\,\#\,T(2,q)\,\#\,T(2,r)$ is nontwisted for any odd positive integers $p,q,r \geq 3$, partially answering in the positive a conjecture of Teragaito [19].

## 1. Introduction

Let $K$ be a knot in the 3-sphere $S^3$, and $D^2$ a disk intersecting $K$ in its interior. Let $n$ be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes $K$ into a new knot $K_n$ in $S^3$. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that $K_n$ is obtained from $K$ by $(n,\omega)$-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\to} K_n$ or $K \overset{(n,\omega)}{\to} K(n,\omega)$. We say that $K_n$ is an $(n,\omega)$-twisted knot (or simply a twisted knot) if $K$ is the unknot (see Figure 1).

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2,3)$ is obtained from the unknot $T(2,1)$ by a $(+1,2)$-twisting (in this case, $n=+1$ and $\omega=+2$). A less obvious example is given in Figure 3, where it is shown that the composite knot $T(2,3)\,\#\,T(2,5)$ can be obtained from the unknot by a $(+1,4)$-twisting (in this case, $n=+1$ and $\omega=+4$); see [10]. Here, $T(2,q)$ denotes the $(2,q)$-torus knot (see [11]).

Active research on twisting of knots started around 1990. One pioneer was the author's Ph.D. thesis advisor Y. Mathieu, who asked the following questions in [13].

Question 1.1. Is every knot in $S^3$ twisted? If not, what is the minimal number of twisting disks?

Question 1.2. Is every twisted knot in $S^3$ prime?

To answer Question 1.1, Miyazaki and Yasuhara [15] were the first to give an infinite family of knots that are nontwisted. In particular, they showed that the granny knot, that is, the product of two right-handed trefoil knots, is the smallest nontwisted knot. In his Ph.D. thesis [3], the author showed that $T(5,8)$ is the smallest nontwisted torus knot. This was followed by a joint work with Yasuhara [4], in which we gave an infinite family of nontwisted torus knots (i.e., $T(p,p+7)$ for any $p \geq 7$) using some techniques derived from old gauge theory. On the other hand, Ohyama [16] showed that any knot in $S^3$ can be untied by (at most) two disks.

Received July 12, 2015. Revision received March 9, 2016.