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Parameterization of the Box Variety by Theta Functions

Eberhard Freitag & Riccardo Salvati Manni

Introduction

We consider the graded algebra (the generators have weight one)

$$B = \mathbb{Q}[Z_1, Z_2, Z_3, W_1, W_2, W_3, C]$$

with defining relations

$$\begin{aligned} W_1^2 + W_2^2 &= Z_3^2, \\ W_1^2 + W_3^2 &= Z_2^2, \\ W_2^2 + W_3^2 &= Z_1^2, \\ W_1^2 + W_2^2 + W_3^2 &= C^2. \end{aligned}$$

This is a normal graded algebra. The associated projective variety $\operatorname{proj}(B)$ is called the box variety or variety of cuboids. It is absolutely irreducible. We denote its complexification by

$$\mathcal{B} := \operatorname{proj}(B \otimes_{\mathbb{Q}} \mathbb{C}).$$

It is a surface that characterizes cuboids. The variables W_i give the edges of the cuboid, the variables Z_i the diagonals of the faces, and C the long diagonal. There is an unsolved problem, raised by Euler, whether the box variety contains nontrivial rational points or not. For more details on the box variety, we refer to [vL] and [ST].

In this note, we describe a parameterization of the box variety by theta functions. This will imply that it is a quotient of the product $\overline{\mathbb{H}/\Gamma[8]} \times \overline{\mathbb{H}/\Gamma[8]}$ of two modular curves of level 8 by a group of order 8 that comes from the diagonal action of $\Gamma[4]$. In fact, this parameterization can be defined over the cyclotomic field $K = \mathbb{Q}(\zeta_8) = \mathbb{Q}(\mathrm{i}, \sqrt{2})$. We found this parameterization from an observation of D. Testa that the box variety can be embedded into a certain Siegel modular variety, which has been described by van Geemen and Nygaard. This background is not necessary for our note, and we will not describe it here. But we want to point out that the still unpublished work [ST] is behind the scenes, and we are very grateful that he explained to us details of this work.

This parameterization can be used to derive quickly known properties and also some new ones of the box variety. For example, we give in Section 2 a modular description of the automorphism group. It can be realized as a subgroup of $\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$. In [ST] and [vL], 140 rational and elliptic curves on the minimal model of the box variety that give the generators of the Picard group have

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