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Volume and Hilbert Function of $\mathbb{R}$-Divisors

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1. Introduction

Let X be a proper, normal algebraic variety of dimension n over a field K, and D an $\mathbb{R}$-divisor on X. The *Hilbert function* of D is the function

$$\mathcal{H}(X, D) : m \mapsto h^0(mD) := \dim_K H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))$$

defined for all $m \in \mathbb{R}$. If D is an ample Cartier divisor, then $\mathcal{H}(X, D)$ agrees with the usual Hilbert polynomial whenever $m \gg 1$ is an integer, but in general $\mathcal{H}(X, D)$ is not a polynomial, not even if D is a $\mathbb{Z}$-divisor and $m \in \mathbb{Z}$. The simplest numerical invariant associated to the Hilbert function is the *volume* of D, defined as

$$\mathrm{vol}(D) := \limsup_{m\to\infty} \frac{h^0(mD)}{m^n/n!}.$$

If E is an effective $\mathbb{R}$-divisor, then

$$h^0(mD - mE) \le h^0(mD) \le h^0(mD + mE) \tag{$*$}$$

for every $m > 0$; hence,

$$\mathrm{vol}(D - E) \le \mathrm{vol}(D) \le \mathrm{vol}(D + E). \tag{$**$}$$

Furthermore, if equality holds in ($*$) for every $m \gg 1$, then equality holds in ($**$). The aim of this note is to prove the converse for *big* divisors, that is, when $\mathrm{vol}(D) > 0$. Although the volume does not determine the Hilbert function, we prove that

$$\begin{aligned} \mathcal{H}(X, D) \equiv \mathcal{H}(X, D - E) \quad &\Leftrightarrow \quad \mathrm{vol}(D) = \mathrm{vol}(D - E) \quad \text{and} \\ \mathcal{H}(X, D) \equiv \mathcal{H}(X, D + E) \quad &\Leftrightarrow \quad \mathrm{vol}(D) = \mathrm{vol}(D + E). \end{aligned}$$

As a byproduct of the proof, we also obtain a characterization of such divisors E in terms of the negative part $N_\sigma(D)$ of the *Zariski–Nakayama-decomposition* (also called σ-decomposition) and of the divisorial part of the *augmented base locus* $\mathbf{B}_+^{\mathrm{div}}(D)$; see [Nak04], (4.1) and Definition 5.1 for definitions.

Another interesting consequence is that the answer depends only on the $\mathbb{R}$-linear equivalence class of D. This is obvious for $\mathbb{Z}$-linear equivalence, but it can easily happen that $D' \sim_{\mathbb{R}} D$ yet $h^0(X, mD) \ne h^0(X, mD')$ for every $m > 0$; see Example 2.6. In fact, the only relationship between $\mathcal{H}(X, D)$ and $\mathcal{H}(X, D')$ that we know of is $\mathrm{vol}(D) = \mathrm{vol}(D')$.

Our main results are the following.

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