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A Topological Characterization of the Underlying Spaces of Complete R-Trees

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Abstract. We prove that a topological space (P, τ) admits a compatible metric d such that (P, d) is a complete R-tree if and only if P is a topological R-tree (i.e. metrizable, locally path-connected, and uniquely arcwise connected) and also *locally interval compact.* The latter notion means that each point $x \in P$ has a closed neighborhood $\overline{U}$ such that $\overline{U} \cap \alpha$ is compact for each closed half interval $\alpha \subset P$. For topological R-trees, the property "locally interval compact" is strictly stronger than topological completeness.

1. Introduction

An *R-tree* (P, d) is a uniquely arcwise connected metric space such that for each pair of points $\{x, y\} \subset P$, the arc $([x, y], d) \subset P$ from x to y is isometric to the Euclidean segment $[0, d(x, y)]$. R-trees have received considerable attention as objects of study in their own right, and R-trees also play a prominent role in geometric group theory, notably in the study of group actions on spaces of nonpositive curvature [1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 14; 15; 16; 17; 18; 20; 21; 22; 23; 24; 25; 26; 29; 30].

However, the following fundamental question has apparently escaped collective inquiry: Which topological spaces (P, τ) underly the complete R-trees?

To answer this question, observe that open metric balls in the metric R-tree (P, d) are path connected and hence (P, τ) is metrizable, uniquely arcwise connected, and locally path connected, that is, R-trees are *topological R-trees.* Thanks to a result of John Mayer and Lex Oversteegen [27], the converse is also true: each topological R-tree (P, τ) is the underlying space of some R-tree (P, d). (A preprint of the author contains an alternate shorter proof [13].)

For the metric R-tree (P, d) to be complete, it is of course necessary that (P, τ) is topologically complete, but somewhat surprisingly, this is not sufficient. Example 1, the planar subspace $([0, 1] \times \{0\}) \cup (\bigcup_{n=1}^{\infty} \{\frac{1}{n}\} \times [0, \frac{1}{n}))$, shows it is *false* that a topologically complete topological R-tree (P, τ) is necessarily the underlying space of some complete R-tree (P, d).

As mentioned in the abstract, to strengthen topological completeness and ensure that the topological R-tree (P, τ) is the underlying space of a complete metric R-tree, it is precisely adequate to demand that (P, τ) has the following extra property:

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