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On the Convergence of Gromov–Witten Potentials and Givental's Formula

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Abstract. Let X be a smooth projective variety. The Gromov–Witten potentials of X are generating functions for the Gromov–Witten invariants of X: they are formal power series, sometimes in infinitely many variables, with Taylor coefficients given by Gromov–Witten invariants of X. It is natural to ask whether these formal power series converge. In this paper we describe and analyze various notions of convergence for Gromov–Witten potentials. Using results of Givental and Teleman, we show that if the quantum cohomology of X is analytic and generically semisimple, then the genus-g Gromov–Witten potential of X converges for all g. We deduce convergence results for the all-genus Gromov–Witten potentials of compact toric varieties, complete flag varieties, and certain noncompact toric varieties.

1. Introduction

Let X be a smooth projective variety. The total descendant potential of X is a generating function for the Gromov–Witten invariants of X. It is a formal power series $\mathcal{Z}_X$ in $\hbar$, $\hbar^{-1}$, and infinitely many variables t_k^α, $0 \le \alpha \le N$, $0 \le k < \infty$, with Taylor coefficients given by Gromov–Witten invariants of X. Here $t_0, t_1, t_2, \ldots$ is an infinite sequence of cohomology classes on X, $t_k = t_k^0 \phi_0 + \cdots + t_k^N \phi_N$ is the expansion of t_k in terms of a basis $\{\phi_\alpha\}$ for $H^\bullet(X)$, and

$$\mathcal{Z}_X = \exp\left(\sum_{g \ge 0} \hbar^{g-1} \mathcal{F}_X^g\right),$$

where $\mathcal{F}_X^g$ is a generating function for genus-g Gromov–Witten invariants. It is known that $\mathcal{Z}_X$ does not converge[1] as a series in $\hbar$ and $\hbar^{-1}$, but it is natural to ask whether the formal power series $\mathcal{F}_X^g$ converge. This question is particularly relevant in light of work by Ruan and his collaborators on Gromov–Witten theory and birational geometry. If $X \dashrightarrow Y$ is a crepant birational map between smooth projective varieties (or orbifolds), then, very roughly speaking, the total descen-

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[1] $\mathcal{Z}_X$ should be regarded as an asymptotic expansion in $\hbar$.