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The Affine Automorphism Group of $\mathbb{A}^3$ is Not a Maximal Subgroup of the Tame Automorphism Group

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Abstract. We construct explicitly a family of proper subgroups of the tame automorphism group of affine three-space (in any characteristic) that are generated by the affine subgroup and a nonaffine tame automorphism. One important corollary is the titular result that settles negatively the open question (in characteristic zero) of whether the affine subgroup is a maximal subgroup of the tame automorphism group. We also prove that all groups of this family have the structure of an amalgamated free product of the affine group and a finite group over their intersection.

1. Introduction

Throughout, $\mathbb{K}$ denotes a field of any characteristic. We denote by $\mathrm{GA}_n(\mathbb{K})$ the group of polynomial automorphisms of $\mathbb{A}^n_{\mathbb{K}}$. We consider $\mathrm{Aff}_n(\mathbb{K})$ (resp. $\mathrm{BA}_n(\mathbb{K})$, resp. $\mathrm{TA}_n(\mathbb{K})$), the subgroup of $\mathrm{GA}_n(\mathbb{K})$ of affine (resp. triangular, resp. tame) automorphisms (see Section 2 or [4] for precise definitions). In this paper we are interested with the question of finding proper intermediate subgroups between $\mathrm{Aff}_n(\mathbb{K})$ and $\mathrm{TA}_n(\mathbb{K})$.

If $n = 2$, then it is well known that such intermediate subgroups exist. The classical Jung–van der Kulk theorem [5; 6] states that $\mathrm{GA}_2(\mathbb{K}) = \mathrm{TA}_2(\mathbb{K})$ and, moreover, $\mathrm{GA}_2(\mathbb{K})$ is the amalgamated free product of $\mathrm{Aff}_2(\mathbb{K})$ and $\mathrm{BA}_2(\mathbb{K})$ along their intersection. Using this structure theorem, we can uniquely define the height of any automorphism $\phi \in \mathrm{GA}_2(\mathbb{K})$ as the maximum of the degrees of the triangular automorphisms in any reduced decomposition of ϕ. Let H_d denote the set of all automorphisms of height at most d. Then we have that $\mathrm{Aff}_2(\mathbb{K}) = H_1 \subset H_2 \subset H_3 \subset \cdots \subset \mathrm{TA}_2(\mathbb{K})$ is an ascending sequence of (proper) subgroups of $\mathrm{TA}_2(\mathbb{K})$. In particular, for all $\beta \in \mathrm{BA}_2(\mathbb{K}) \setminus \mathrm{Aff}_2(\mathbb{K})$, $\langle \mathrm{Aff}_2(\mathbb{K}), \beta\rangle$ is a proper subgroup of $\mathrm{TA}_2(\mathbb{K})$.

In the case that $n > 2$ and $\mathbb{K}$ has positive characteristic, it is also known that there are many intermediate subgroups between $\mathrm{Aff}_n(\mathbb{K})$ and $\mathrm{TA}_n(\mathbb{K})$ (see, e.g., [3]). However, in characteristic zero, the question is much more nuanced.[1] The first partial results in this direction concern subgroups of the form $\langle \mathrm{Aff}_n(\mathbb{K}), \beta\rangle$

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[1] Recently, Wright [8] showed that in characteristic zero, $\mathrm{TA}_3(\mathbb{K})$ is an amalgamated free product of three subgroups along their pairwise intersection, which implies a much weaker structure on $\mathrm{TA}_3(\mathbb{K})$. Unlike in dimension two, we no longer have a reasonably unique representation of every tame automorphism.