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The Complex Structure of the Teichmüller Space

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Abstract. The Teichmüller space of a topological surface X is a space that parameterizes complex structures on X up to the action of homeomorphisms that are isotopic to the identity. This space itself has a complex structure defined in terms of Beltrami differentials and quasi-conformal mappings. For X a surface of genus g and m punctures, n geodesics $A_1, \dots, A_n$ ($n = 6g-6+2m$) can be chosen so that their hyperbolic translation lengths $(L(A_1), \dots, L(A_n))$ give a local parameterization of the Teichmüller space.

In this paper we describe the almost complex structure as a real matrix acting on the tangent space with basis $(\partial/\partial L(A_1), \dots, \partial/\partial L(A_n))$. In the cotangent space the natural Hermitian scalar product of the associated quadratic differentials $(\Theta_{A_1}, \dots, \Theta_{A_n})$ determines a skew-symmetric real matrix C and a symmetric matrix S. We prove that the matrix of the complex structure is SC^{-1}.

Introduction

The Teichmüller space of a topological surface X is a space that parameterizes complex structures on X up to the action of homeomorphisms that are isotopic to the identity. This space itself has a complex structure, which is defined in terms of Beltrami differentials and quasi-conformal mappings. We describe the relationship of this complex structure in terms of the variation of the lengths of geodesics on a variable surface X_τ. We will view such surfaces as a quotient space of the upper half-plane factored by variable fixed-point-free Fuchsian groups Γ_τ. The upper half-plane has a hyperbolic metric whose corresponding geodesics are semicircles and half-lines orthogonal to $\mathbb{R}$.

A hyperbolic element A in Γ has a unique geodesic axis and a well-defined translation length $L(A)$ where

$$\cos h\left(\frac{L(A)}{2}\right) = \frac{1}{2}|\operatorname{trace}(A)|.$$

For X a surface of genus g and m punctures, the group elements of Γ are expressed as real analytic functions of finitely many group elements $A_1, \dots, A_n$ ($n = 6g-6+2m$); and the n-tuple $(L(A_1), \dots, L(A_n))$ gives a local coordinate chart of the Teichmüller space. (For these classical facts, see, e.g., Gardiner [2, pp. 153–157], Ahlfors [1].)

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