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Higher Derivatives of Length Functions along Earthquake Deformations

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1. Introduction

Let S be a closed surface of genus $g \geq 2$, and $T(S)$ the associated Teichmüller space of hyperbolic structures on S. Given $\gamma \in \pi_1(S)$, let $L_\gamma : T(S) \to \mathbb{R}$ be the associated length function, and $T_\gamma : T(S) \to \mathbb{R}$ the associated trace function. The functions L_γ, T_γ have a simple relation given by

$$T_\gamma = 2\cosh(L_\gamma/2). \tag{1}$$

Let β be the homotopy class of a simple multicurve (i.e., a union of disjoint simple nontrivial closed curves in S), and t_β the vector field on $T(S)$ associated with left twist along the geodesic representative of β (see [4]). In this paper, we describe a formula to calculate the higher-order derivatives of the functions L_γ, T_γ along t_β. In particular, we will find a formula for

$$t_\beta^k L_\gamma = t_\beta t_\beta \dots t_\beta L_\gamma.$$

The formulae we derive generalize formulae for the first two derivatives derived by Kerchoff [4] (first derivative) and Wolpert [5; 6] (first and second derivatives).

Kerckhoff and Wolpert both showed that the first derivative is given by

$$t_\beta L_\gamma = \sum_{p \in \beta' \cap \gamma'} \cos\theta_p, \tag{2}$$

where β', γ' are the geodesic representatives of β, γ, respectively, and θ_p is the angle of intersection at $p \in \beta' \cap \gamma'$. Kerckhoff [4] further generalized the formula for the case where β, γ are measured laminations.

Wolpert [6] derived the following formula for the second derivative:

$$\begin{aligned} t_\alpha t_\beta L_\gamma = {} & \sum_{(p,q) \in \beta' \cap \gamma' \times \alpha' \cap \gamma'} \frac{e^{l_{pq}} + e^{l_{qp}}}{2(e^{L_\gamma} - 1)} \sin\theta_p \sin\theta_q \\ & + \sum_{(r,s) \in \beta' \cap \gamma' \times \beta' \cap \alpha'} \frac{e^{m_{rs}} + e^{m_{sr}}}{2(e^{L_\beta} - 1)} \sin\theta_r \sin\theta_s, \end{aligned}$$

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