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# The Trace Map of Frobenius and Extending Sections for Threefolds

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Abstract. In this paper, by using the trace map of Frobenius, we consider problems on extending sections for positive characteristic threefolds.

## 0. Introduction

In characteristic zero, by the Kodaira vanishing theorem and its generalizations, we can establish some results on adjoint divisors, such as the Kawamata–Shokurov basepoint-free theorem (see, e.g., [Kollár–Mori, Theorem 3.3]) and the Hacon–McKernan extension theorem [HM, Theorem 5.4.21]. These theorems claim, under suitable conditions, that an adjoint divisor $m(K_X+\Delta+A)$ has good properties, where $m\in\mathbb{Z}_{>0}$, $(X,\Delta)$ is a pair, and $A$ is an ample divisor. In this paper we only consider the following very simple situation: $X$ is a smooth projective variety, $\Delta=S$ is a smooth prime divisor, and $A$ is an ample Cartier divisor. The following fact immediately follows from the Kodaira vanishing theorem.

Fact 0.1. *Let $k$ be an algebraically closed field of characteristic zero. Let $X$ be a smooth projective variety over $k$. Let $S$ be a smooth prime divisor on $X$, and let $A$ be an ample Cartier divisor on $X$ such that $K_X+S+A$ is nef. Fix $m\in\mathbb{Z}_{>0}$. Then, by the Kodaira vanishing theorem, we obtain*

$$H^1(X,K_X+A+(m-1)(K_X+S+A))=0.$$

*Thus, the natural restriction map*

$$H^0(X,m(K_X+S+A))\to H^0(S,m(K_S+A|_S))$$

*is surjective.*

It is natural to consider whether this fact also holds in positive characteristic. Unfortunately, however, there exists the following example.

Example 0.2 (cf. Example 4.4). Let $k$ be an algebraically closed field of positive characteristic. Then, there exist a smooth projective surface $X$ over $k$, a smooth prime divisor $C$ on $X$, and an ample Cartier divisor $A$ on $X$ such that $K_X+C+A$ is nef and the natural restriction map

$$H^0(X,K_X+C+A)\to H^0(C,K_C+A|_C)$$

is not surjective.

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