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On Factoriality of Threefolds with Isolated Singularities

Francesco Polizzi, Antonio Rapagnetta, &
Pietro Sabatino

Abstract. We investigate the existence of complete intersection threefolds $X \subset \mathbb{P}^n$ with only isolated, ordinary multiple points and we provide some sufficient conditions for their factoriality.

0. Introduction

Grothendieck–Lefschetz's theorem ([Gro68, Exposé XII, Corollaire 3.7; Har70, Chapter IV, Corollary 3.3; BS95, Corollary 2.3.4]) says that if X is an effective, ample divisor of a smooth variety Y defined over a field of characteristic 0, then the restriction map of Picard groups

$$\operatorname{Pic} Y \longrightarrow \operatorname{Pic} X$$

is injective if $\dim X \geq 2$ and is an isomorphism if $\dim X \geq 3$. One might ask what happens, with the same hypotheses, to the restriction map $\mathrm{CH}^1(Y) \to \mathrm{CH}^1(X)$ between rational equivalence classes of codimension 1 subvarieties. Under some mild assumptions on the singularities of X (e.g., if X is normal), this is equivalent to asking whether or not the conclusions of Grothendieck–Lefschetz's theorem for Picard groups remain true for the restriction map

$$\operatorname{Cl} Y \longrightarrow \operatorname{Cl} X, \tag{1}$$

where, as usual, $\operatorname{Cl} X$ denotes the *class group* of X, namely the group of linear equivalence classes of Weil divisors. When X is smooth, there is nothing new to say since the groups $\operatorname{Pic} X$ and $\operatorname{Cl} X$ are isomorphic; however, when X is singular, the problem becomes a delicate one.

We will restrict ourselves to the case where $Y \subset \mathbb{P}^n$ ($n \geq 4$) is a smooth, complete intersection fourfold and $X \subset Y$ is a threefold with isolated singularities. Since X is projectively normal and nonsingular in codimension 1, the map (1) is an isomorphism precisely when $\operatorname{Pic} X = \operatorname{Cl} X = \mathbb{Z}$, generated by the class of $\mathcal{O}_X(1)$. This is in turn equivalent to the fact that the homogeneous coordinate ring of X is a UFD or that any hypersurface in X is the complete intersection of X with a hypersurface of $\mathbb{P}^n$. In this case we say that X is *factorial.*

In the recent years, the study of factoriality of threefolds in $\mathbb{P}^4$ having only ordinary double points ("nodes") has attracted the attention of several authors. In particular, the following result was conjectured, and proven in a weaker form, by Ciliberto and Di Gennaro ([CDG04a]). The proof of the general case is due to Cheltsov ([Che10b; Che10a]).

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