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The Additive Problem with One Cube and Three Cubes of Primes

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Abstract. In this paper, we establish that all positive integers up to N but at most $O(N^{25/27+\varepsilon})$ exceptions can be represented as the sum of a cube and three cubes of primes. This improves upon the earlier result $O(N^{17/18+\varepsilon})$ obtained by Ren and Tsang [4].

1. Introduction

In 1949, Roth [5] investigated the expression of positive integers n as the sum of a cube and three cubes of primes, that is,

$$n = x^3 + p_1^3 + p_2^3 + p_3^3, \tag{1.1}$$

where x is a positive integer, and p_1, p_2, p_3 are primes. The philosophy of the Hardy–Littlewood circle method suggests that every sufficiently large integer n can be expressed in the form (1.1). Roth [5] proved that almost all positive integers n can be written as (1.1). In order to introduce Roth's theorem more precisely, we denote by $r(n)$ the number of representations of n in the form (1.1) and define

$$E(N) = |\{1 \le n \le N : \ r(n) = 0\}|. \tag{1.2}$$

Roth's theorem actually states that $E(N) \ll N \log^{-A} N$ for arbitrary large constant $A > 0$. Roth's theorem has been refined by Ren [2] to

$$E(N) \ll N^{169/170}. \tag{1.3}$$

Recently, further improvement has been obtained in a series of papers by Ren and Tsang [3; 4]. In particular, it was proved in [3] that $E(N) \ll N^{1{,}271/1{,}296+\varepsilon}$, and it was established in [4] that

$$E(N) \ll N^{17/18+\varepsilon}. \tag{1.4}$$

In this paper, we establish the following result.

Theorem 1.1. *Let $E(N)$ be defined in (1.2). Then for any $\varepsilon > 0$, we have*

$$E(N) \ll N^{25/27+\varepsilon}. \tag{1.5}$$

We establish Theorem 1.1 by the Hardy–Littlewood circle method. We employ the technique developed by Vaughan [6; 7]. This technique was recently used by Koichi Kawada to prove that all large even integers can be written as the sum of seven cubes of primes and a cube with at most two prime factors. In prior

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