Michigan Math. J. 63 (2014), 535–552

Three Techniques for Obtaining Algebraic Circle Packings

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Abstract. The main purpose of this article is to demonstrate three techniques for proving algebraicity statements about circle packings. We give proofs of three related theorems: (1) that every finite simple planar graph is the contact graph of a circle packing on $\hat{\mathbb{C}}$, equivalently in $\mathbb{C}$, all of whose tangency points, centers, and radii are algebraic, (2) that every flat conformal torus that admits a circle packing whose contact graph triangulates the torus has algebraic modulus, and (3) that if R is a compact Riemann surface of genus at least 2, having constant curvature -1 and admitting a circle packing whose contact graph triangulates R, then R is isomorphic to the quotient of $\mathbb{H}^2$ by a subgroup of $\mathrm{PSL}_2(\mathbb{R} \cap \bar{\mathbb{Q}})$. The statement (1) is original, whereas (2) and (3) have been previously proved in [McC96, Chapters 8, 9], the Ph.D. thesis of McCaughan.

Our first proof technique is to apply Tarski's theorem, a result from model theory, which says that if an elementary statement in the theory of real-closed fields is true over one real-closed field, then it is true over any real closed field. This technique works to prove (1) and (2). Our second proof technique is via an algebraicity result of Thurston on finite covolume discrete subgroups of $\mathrm{PSL}_2\,\mathbb{C} \subset \mathrm{Isom}\,\mathbb{H}^3$. This technique works to prove (1). Our first and second techniques had not previously been applied in this area. Our third and final technique is via a lemma in real algebraic geometry, and was previously used by McCaughan to prove (2) and (3). We show that in fact it may be used to prove (1) as well.

1. Introduction

A *circle packing* in $\hat{\mathbb{C}}$ is defined to be a finite collection of pairwise interiorwise disjoint metric closed disks in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ equipped with the constant curvature $+1$ metric as usual. There are no conditions on the radii of the disks. We say that two closed disks are *tangent* if their boundary circles are tangent. The *contact graph* of a circle packing $\mathcal{P}$ is the graph G whose vertex set

Received August 9, 2013. Revision received December 22, 2013.

The second author has been partially supported by NSF grants DMS-0456940, DMS-0555750, DMS-0801029, DMS-1101373. The third author has been partially supported by NSF grant DMS-0706878 and the Alfred P. Sloan Foundation.