Michigan Math. J. 63 (2014), 253–273

Dirichlet Series Associated to Cubic Fields with Given Quadratic Resolvent

Henri Cohen & Frank Thorne

Abstract. Let k be a quadratic field. We give an explicit formula for the Dirichlet series $\sum_K |\operatorname{Disc}(K)|^{-s}$, where the sum is over isomorphism classes of all cubic fields whose quadratic resolvent field is isomorphic to k.

Our work is a sequel to [14] (see also [22]), where such formulas are proved in a more general setting, in terms of sums over characters of certain groups related to ray class groups. In the present paper we carry the analysis further and prove explicit formulas for these Dirichlet series over $\mathbb{Q}$, and in a companion paper we do the same for quartic fields having a given cubic resolvent.

As an application, we compute tables of the number of S_3-sextic fields E with $|\operatorname{Disc}(E)| < X$ for X ranging up to 10^{23}. An accompanying PARI/GP implementation is available from the second author's website.

1. Introduction

A classical problem in algebraic number theory is that of *enumerating number fields* by discriminant. Let $N_d^{\pm}(X)$ denote the number of isomorphism classes of number fields K with $\deg(K) = d$ and $0 < \pm\operatorname{Disc}(K) < X$. The quantity $N_d^{\pm}(X)$ has seen a great deal of study; see (e.g.) [12; 4; 30] for surveys of classical and more recent work.

It is widely believed that $N_d^{\pm}(X) = C_d^{\pm}X + o(X)$ for all $d \geq 2$. For $d = 2$, this is classical, and the case $d = 3$ was proved in 1971 work of Davenport and Heilbronn [19]. The cases $d = 4$ and $d = 5$ were proved much more recently by Bhargava [3; 6]. In addition, Bhargava [5] also conjectured a value of the constants $C_{d,S_d}^{\pm}$ for $d > 5$, where the additional index S_d means that one counts only degree d number fields with Galois group of the Galois closure isomorphic to S_d.

Related questions have also seen recent attention. For example, Belabas [1] developed and implemented a fast algorithm to compute large tables of cubic fields, which has proved essential for subsequent numerical computations (including one to be carried out in this paper!). Based on Belabas's data, Roberts [26] conjectured the existence of a secondary term of order $X^{5/6}$ in $N_3^{\pm}(X)$, and this was proved (independently and using different methods) by Bhargava, Shankar, and

Received January 15, 2013. Revision received March 1, 2014.

The second author was partially supported by the National Science Foundation under Grant DMS-0802967.