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Quantum Schubert Cells via Representation Theory and Ring Theory

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1. Introduction

The study of the spectra of quantum groups for generic deformation parameters was initiated twenty years ago by Joseph [20; 21] and Hodges–Levasseur–Toro [18] who obtained a number of important results on them. One of the long-term goals was to understand these spectra geometrically in terms of symplectic foliations in an attempt to extend the orbit method [9] to more general classes of algebras and Poisson manifolds. This grew into a very active area of studying the ring theoretic properties of quantum analogues of universal enveloping algebras of solvable Lie algebras. The quantum Schubert cell algebras, defined by De Concini–Kac–Procesi [8] and Lusztig [25], comprise one of the major families of algebras in this area. There is one such algebra $\mathcal{U}^-[w]$ for every simple Lie algebra $\mathfrak{g}$ and an element w of the Weyl group W of $\mathfrak{g}$. It is a subalgebra of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ and a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_- \cap w(\mathfrak{n}_+))$, where $\mathfrak{n}_\pm$ are the nilradicals of a pair of opposite Borel subalgebras $\mathfrak{b}_\pm$ of $\mathfrak{g}$. From another perspective, the algebra $\mathcal{U}^-[w]$ is a deformation of the coordinate ring of the Schubert cell corresponding to w of the full flag variety of $\mathfrak{g}$, equipped with the standard Poisson structure [14]. These algebras played important roles in many different contexts in recent years such as the study of coideal subalgebras of $\mathcal{U}_q(\mathfrak{b}_-)$ and $\mathcal{U}_q(\mathfrak{g})$ [17; 16] and quantum cluster algebras [10].

There are two very different approaches to the study of the spectra of $\mathcal{U}^-[w]$. One is purely ring theoretic and is based on the Cauchon procedure of deleting derivations [6]. The second is a representation theoretic one and builds on the above mentioned methods of Joseph, Hodges, Levasseur, and Toro [21; 18]. Each of these methods has a number of advantages over the other, and relating them is an important open problem with many potential applications. Previously there were no connections between them even for special cases of the algebras $\mathcal{U}^-[w]$, such as the algebras of quantum matrices.

In this paper we unify the ring theoretic and the representation theoretic approaches to the study of Spec$\,\mathcal{U}^-[w]$. Furthermore, we resolve several other open problems on the deleting derivation procedure and the spectra of $\mathcal{U}^-[w]$, the two being questions posed by Cauchon and Mériaux [27]. Before we proceed with the statements of these results, we need to introduce some additional background.

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