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Euler–Mellin Integrals and A-Hypergeometric Functions

Christine Berkesch, Jens Forsgård, & Mikael Passare†

Abstract. We consider integrals that generalize both Mellin transforms of rational functions of the form $1/f$ and classical Euler integrals. The domains of integration of our so-called Euler–Mellin integrals are naturally related to the coamoeba of f, and the components of the complement of the closure of this coamoeba give rise to a family of these integrals. After performing an explicit meromorphic continuation of Euler–Mellin integrals, we interpret them as A-hypergeometric functions and discuss their linear independence and relation to Mellin–Barnes integrals.

1. Introduction

In the classical theory of hypergeometric functions, a prominent role is played by the Euler integral formula

$$ {}_2F_1(s;t;u) = \frac{\Gamma(t)}{\Gamma(s_1)\Gamma(s_2)} \int_0^1 x^{s_1-1}(1-x)^{t-s_1-1}(1-ux)^{-s_2}\,dx, $$

which yields an analytic continuation of the Gauss hypergeometric series ${}_2F_1$ from the unit disk $|u| < 1$ to the larger domain $|\arg(1-u)| < \pi$. However, this Euler integral is not symmetric in s_1 and s_2, even though the function ${}_2F_1$ enjoys such symmetry. Following Erdélyi [Erd37], we can introduce another variable of integration and obtain the symmetric formula

$$ \begin{aligned} {}_2F_1(s;t;u) = G(s,t) \int_0^1 \int_0^1 & x^{s_1-1} y^{s_2-1}(1-x)^{t-s_1-1} \\ & \times (1-y)^{t-s_2-1}(1-uxy)^{-t}\,dx \wedge dy, \end{aligned} $$

$$ \text{where } G(s,t) = \frac{\Gamma(t)^2}{\Gamma(s_1)\Gamma(s_2)\Gamma(t-s_1)\Gamma(t-s_2)}. \tag{1.1} $$

After making the substitutions $z = x/(1-x)$, $w = y/(1-y)$, and $c = 1-u$, we find that the double integral in (1.1) takes the simple form

$$ \int_0^\infty \int_0^\infty \frac{z^{s_1} w^{s_2}}{(1+z+w+czw)^t} \frac{dz \wedge dw}{zw}, \tag{1.2} $$

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