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Proper Holomorphic Mappings on Flag Domains of SU(p, q) Type on Projective Spaces

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1. Introduction

The objects of study in this paper are the domains in $\mathbb{P}^n$ defined by

$$\mathbb{D}_n^\ell = \left\{[z_0, \dots, z_n] \in \mathbb{P}^n : \sum_{j=0}^{\ell} |z_j|^2 > \sum_{j=\ell+1}^{n} |z_j|^2\right\}$$

and the proper holomorphic mappings among them. They are examples of the so-called *flag domains* in $\mathbb{P}^n$ when the latter is regarded as a flag variety. More explicitly, they are open orbits of the real forms $\mathrm{SU}(\ell+1, n-\ell)$ of the complex simple Lie group $\mathrm{SL}(n+1, \mathbb{C})$ when both of which act on $\mathbb{P}^n$ as biholomorphisms.

The domain $\mathbb{D}_n^0$ is just the complex unit n-ball embedded in $\mathbb{P}^n$, and there has been an extensive literature in the study of their proper holomorphic mappings in the last couple of decades. For a survey, see [Fo]. In general, when the codimension is high, the set of proper holomorphic mappings between complex unit balls is large and difficult to determine. On the other hand, in [BH] and [BEH] the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$ and the associated holomorphic mappings are studied by methods in Cauchy–Riemann geometry. It appears that there is in general much more rigidity when $\ell \geq 1$. Indeed, there is one essential difference between the complex unit n-ball and the domains $\mathbb{D}_n^\ell$ with $\ell \geq 1$, for the latter contain linear subspaces of $\mathbb{P}^n$. Motivated by this, the author of this paper studied in [N] the domains $\mathbb{D}_n^\ell$, $\ell \geq 1$, and their generalizations in Grassmannians by exploiting the structure of the moduli spaces of compact complex analytic subvarieties. Rigidity results analogous to those of [BH] are obtained in a more geometric way.

We will follow the terminology in [BH; BEH] and call ℓ the *signature* of the domain $\mathbb{D}_n^\ell$. As far as rigidity of holomorphic mappings among those domains is concerned, the determining factor should be the difference in signatures rather than the codimension. This is illustrated in [BH], for instance, where it is shown that if the domain and target are of the same signature then any local proper holomorphic map is the restriction of a linear embedding between the ambient projective spaces. On the other hand, Baouendi, Ebenfelt, and Huang [BEH] studied the situations with a small signature difference. Together with other results, they proved that there is partial rigidity for local proper holomorphic mappings $h : U \subset \mathbb{D}_n^\ell \to \mathbb{D}_m^{\ell'}$

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