Michigan Math. J. 62 (2013), 705–720

Cohomology of Local Systems on Loci of d-elliptic Abelian Surfaces

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1. Introduction

To an irreducible representation of Sp_{2g} with highest weight vector λ, one can associate in a natural way a local system $\mathbf{W}_\lambda$ on the moduli spaces $\mathcal{A}_g$ and hence also on $\mathcal{M}_g$. One reason for studying these local systems is that their complex (resp., ℓ-adic) cohomology groups will contain spaces of elliptic and Siegel modular forms (resp., their associated ℓ-adic Galois representations) as subquotients. In particular, one can study modular forms by looking at the cohomology of these local systems—and vice versa.

When $g = 1$ this is described by the Eichler–Shimura theory and in particular by its Hodge-theoretic/ℓ-adic interpretation [8], which expresses the cohomology of such a local system in terms of spaces of modular forms on the corresponding modular curve. See [12, Sec. 4] for a summary. For higher genera the situation is not as well understood. The (integer-valued) Euler characteristics of these local systems on $\mathcal{M}_2$ were calculated in [17]. Their Euler characteristics on $\mathcal{M}_g$ and $\mathcal{A}_g$ for $g = 2, 3$, now taken in the Grothendieck group of ℓ-adic Galois representations, have been investigated by means of point counting in the sequence of papers [10; 11; 2; 3].

Another reason to be interested in such local systems is that they arise when computing the cohomology of relative configuration spaces. For instance, in the case of $\mathcal{M}_g$, the results of [16] imply that calculating the Euler characteristics of all of these local systems on $\mathcal{M}_g$ is equivalent to calculating the $\mathbb{S}_n$-equivariant Euler characteristic of $\mathcal{M}_{g,n}$ for all n.

In this paper, we shall study the restriction of these local systems to certain loci in $\mathcal{A}_2$ of abelian surfaces with a degree d^2 isogeny to a product of elliptic curves. We call such surfaces d-elliptic and denote the (normalization of the) locus of d-elliptic surfaces by $\mathcal{E}_d$. A curve of genus 2 is d-elliptic in the usual sense—that is, it admits a degree d covering onto an elliptic curve if and only if its Jacobian is d-elliptic in this sense. Already the simplest case $d = 1$ (i.e., the locus $\mathcal{A}_2 \setminus \mathcal{M}_2$ of products of elliptic curves) is not entirely trivial, as one needs the branching formula of Section 3.

Received June 16, 2012. Revision received September 13, 2012.

The author is supported by the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine.