Michigan Math. J. 62 (2013), 631–656

Transference of Density

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1. Introduction and Notation

Let $H = \{(x, y) : y \geq 0\}$ denote the upper half-plane. This paper concerns various linear densities of a set $E \subset H$ at points of $\mathbb{R}$, which we identify with the boundary of H.

We shall denote by $L(x, \theta)$ the ray $\{(x + t\cos\theta, t\sin\theta) : t \geq 0\}$ for every $x \in \mathbb{R}$ and $\theta \in (0, \pi)$. The segment $\{(x + t\cos\theta, t\sin\theta) : 0 \leq t \leq r\}$ will be denoted by $L(x, \theta, r)$. The density of E along the ray $L(x, \theta)$ is defined by

$$d(E, x, \theta) = \lim_{r\to 0+} \frac{\lambda(E \cap L(x, \theta, r))}{r}, \tag{1}$$

where λ denotes the linear measure (one-dimensional Hausdorff measure) in $\mathbb{R}^2$. Replacing the limit in (1) by lim sup and lim inf, we obtain the respective upper and lower densities $\bar{d}(E, x, \theta)$ and $\underline{d}(E, x, \theta)$. Should E be non-Borel, there are several additional possibilities defined by replacing λ in (1) with either the linear outer measure λ^* or the linear inner measure λ_* and again replacing the limit by either lim sup and lim inf. So, for example, the upper inner density of E along the ray $L(x, \theta)$ is defined as

$$\bar{d}_*(E, x, \theta) = \limsup_{r\to 0+} \frac{\lambda_*(E \cap L(x, \theta, r))}{r}.$$

If $d\#$ denotes any of these density operators, then the set E is said to have positive density relative to $d\#$ at a point $x \in \mathbb{R}$ if $d\#(x) > 0$.

In this paper we are interested in whether linear densities in one sense or another are transferable. For example, if we know that a set E has one of these linear densities in a given direction, can we infer that there are points at which E has a linear density of the same or different variety in another direction? The strongest hypothesis for linear densities would be that a set E has full linear density in a given direction and at every $x \in \mathbb{R}$, and the weakest conclusion is that there is a point $x_0 \in \mathbb{R}$ and a direction θ_0 at which $\bar{d}^*(E, x_0, \theta_0) > 0$.

If $E \subset H$ is Borel, then we denote by $D(E, x)$ the two-dimensional density of E at the point $(x, 0)$ relative to H. That is,

$$D(E, x) = \lim_{h\to 0+} \frac{\lambda_2(E \cap B(x, h))}{\lambda_2(H \cap B(x, h))},$$

Received July 16, 2012. Revision received April 23, 2013.