Michigan Math. J. 62 (2013), 475–489

Log-terminal Smoothings of Graded Normal Surface Singularities

Jonathan Wahl

Introduction

Let $(X, 0)$ be the germ of an isolated complex normal singularity. Suppose that its canonical module K_X is $\mathbb{Q}$-Cartier; thus, the canonical sheaf $K_{X-\{0\}}$ has some finite order m. The *index 1* (or *canonical*) cover $(T, 0) \to (X, 0)$ is obtained by normalizing the corresponding cyclic cover. Note that $(T, 0)$ has an isolated normal singularity with K_T Cartier. If T is Cohen–Macaulay, then it is Gorenstein; we call such an $(X, 0)$ $\mathbb{Q}$-*Gorenstein*. (Warning: some authors require only that X be Cohen–Macaulay with K_X $\mathbb{Q}$-Cartier.) That T is Cohen–Macaulay is automatic if X has dimension 2 but not in general, even if X itself is Cohen–Macaulay. In fact, Singh [16, 6.1] constructs an example of an isolated 3-dimensional *rational* (hence Cohen–Macaulay) singularity X with K_X $\mathbb{Q}$-Cartier whose index 1 cover is not Cohen–Macaulay.

Now suppose $(X, 0)$ is a $\mathbb{Q}$-Gorenstein normal surface singularity (e.g., a rational singularity). We will say that a smoothing $f : (\mathcal{X}, 0) \to (\mathbb{C}, 0)$ of X is $\mathbb{Q}$-*Gorenstein* if it is the quotient of a smoothing of the index 1 cover of X. The basic examples are certain smoothings of the cyclic quotient singularities of type $rn^2/rnq - 1$, first mentioned in [8, (5.9)]. In this particular case, $\mathcal{X}$ is a cyclic quotient of $\mathbb{C}^3$—it is even a *terminal* singularity—and a cyclic quotient by a smaller group is the total space of the smoothing of the index 1 cover of X (which is an A_{rn-1}-singularity). The importance of $\mathbb{Q}$-Gorenstein smoothings was first noticed in the work of Kollár and Shepherd-Barron [7]. But even if the (now 3-dimensional) total space $\mathcal{X}$ of a smoothing of X has $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier, it does not immediately follow that the smoothing is $\mathbb{Q}$-Gorenstein.

The main point of this work is that, in an important special case, one can deduce that a smoothing is $\mathbb{Q}$-Gorenstein by proving the much stronger result that the total space $\mathcal{X}$ can be chosen to be *log-terminal*. The result is surprising since the original singularities X are generally not even log-canonical (i.e., have discrepancies $-\infty$).

In the early 1980s the author constructed many examples (both published and unpublished) of surface singularities that admit smoothings with Milnor number 0—that is, smoothings whose Milnor fibre is a $\mathbb{Q}$HD (rational homology disk).

Received April 11, 2012. Revision received January 25, 2013.
Research supported under NSA Grant no. H98230-10-1-0365.