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Ideal-adic Semi-continuity of Minimal Log Discrepancies on Surfaces

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Following Kollár [4], de Fernex, Ein, and Mustaţă [1] proved the ideal-adic semi-continuity of log canonicity effectively, to obtain Shokurov's [6] ACC conjecture for log canonical thresholds on smooth varieties. Mustaţă formulated this semi-continuity for minimal log discrepancies as follows.

Conjecture 1 (Mustaţă; see [3]). *Let (X, Δ) be a pair, Z a closed subset of X, and $\mathcal{I}_Z$ the ideal sheaf of Z. Let $\mathfrak{a} = \prod_{j=1}^{k} \mathfrak{a}_j^{r_j}$ be a formal product of ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. Then there exists an integer l such that the following holds: if $\mathfrak{b} = \prod_{j=1}^{k} \mathfrak{b}_j^{r_j}$ satisfies $\mathfrak{a}_j + \mathcal{I}_Z^l = \mathfrak{b}_j + \mathcal{I}_Z^l$ for all j, then*

$$\mathrm{mld}_Z(X, \Delta, \mathfrak{a}) = \mathrm{mld}_Z(X, \Delta, \mathfrak{b}).$$

The case of minimal log discrepancy 0 is the semi-continuity of log canonicity. Conjecture 1 is proved in the Kawamata log terminal (klt) case in [3, Thm. 2.6]. However, log canonical (lc) singularities are inevitably treated in the study of limits of singularities in the ideal-adic topology because the limit of klt singularities is lc in general. For example, the limit of klt pairs $(\mathbb{A}^2_{x,y}, (x, y^n)\mathcal{O}_{\mathbb{A}^2})$ indexed by $n \in \mathbb{N}$ is the lc pair $(\mathbb{A}^2, x\mathcal{O}_{\mathbb{A}^2})$ in the $(x, y)\mathcal{O}_{\mathbb{A}^2}$-adic topology. The purpose of this paper is to settle Mustaţă's conjecture for surfaces.

Theorem 2. *Conjecture 1 holds when X is a surface.*

We must handle a non-klt triple $(X, \Delta, \mathfrak{a})$ that has positive minimal log discrepancy; yet unlike in the klt case, the log canonicity is not retained when the exponent of $\mathfrak{a}$ is increased as $\mathfrak{a}^{1+\varepsilon}$. For surfaces, however, we are reduced to the purely log terminal (plt) case in which $\mathfrak{a}$ has an expression $\mathfrak{a}'\mathcal{O}_X(-C)$; then we can increase only the exponent of the part $\mathfrak{a}'$ to apply the result on log canonicity.

We work over an algebraically closed field of characteristic 0. We use the notation described next for singularities in the minimal model program.

Notation 3. A *pair* (X, Δ) consists of a normal variety X and an effective $\mathbb{R}$-divisor Δ such that $K_X + \Delta$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We treat a *triple* $(X, \Delta, \mathfrak{a})$ by attaching a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ with positive real exponents r_j. A prime divisor E on a normal variety X' with a

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