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An Extension Theorem for Real Kähler Submanifolds in Codimension 4

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1. Introduction

Submanifold theory, and especially the study of Riemannian submanifolds in Euclidean spaces, are a classic subarea in differential geometry. The Nash embedding theorem [18] guarantees that any complete Riemannian manifold can be isometrically embedded into a Euclidean space. There are many important developments in submanifold theory, of which we mention just two. One is the work of Hongwei Xu and his collaborators [20; 21; 22] generalizing the differentiable sphere theorem of Brendle and Schoen [1; 2] to the submanifold case in order to obtain the optimal pinching constant. The other development we mention here is the work of Marques and Neves [17] in solving the long-standing Willmore conjecture.

Yet in the special case when the submanifold happens to be Kähler, the research is relatively sparse and sporadic, and we believe that the state of knowledge is still rather primitive. In this paper, we shall refer to a Kähler manifold that is isometrically embedded in a real Euclidean space as a *real Kähler Euclidean submanifold,* or *real Kähler submanifold* for short. That is, we have an isometric embedding $f: M^n \to \mathbb{R}^{2n+p}$ from a Kähler manifold M^n of complex dimension n into the real Euclidean space.

Because M^n is equipped with a complex structure, it would be ideal for the embedding f to be both isometric and holomorphic. However, the thesis of Calabi [3] established that very few Kähler metrics can be isometrically and holomorphically embedded in a complex Euclidean space or in other complex space forms. In fact, Calabi precisely characterized all such metrics. So to study generic Kähler manifolds in the extrinsic setting, one must abandon the holomorphicity assumption on the embedding and assume only that it is isometric.

For a real Kähler submanifold $f: M^n \to \mathbb{R}^{2n+p}$, the Kählerness of M^n imposes strong restrictions and makes M^n extremely sensitive to its codimension. For instance, when $p = 1$ (i.e., when M^n is a hypersurface) a result of Florit and Zhang [15] states that, if M^n is also assumed to be complete, then f must be the product of g and the identity map of $\mathbb{C}^{n-1}$; here $g: \Sigma \to \mathbb{R}^3$ is the isometric embedding of a complete surface, which is always Kähler. In other words, surfaces in $\mathbb{R}^3$ are essentially the only real Kähler submanifolds in codimension 1. In contrast, there are all kinds of real hypersurfaces in Euclidean spaces.

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