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Four-manifolds Admitting Hyperelliptic Broken Lefschetz Fibrations

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1. Introduction

A broken Lefschetz fibration is a smooth map from a four-manifold to a surface that has at most two types of singularities: Lefschetz singularity and indefinite fold singularity. This fibration was introduced in [1] as a fibration structure compatible with near-symplectic structures.

A simplified broken Lefschetz fibration is a broken Lefschetz fibration over the sphere that satisfies several conditions on fibers and singularities. This fibration was first defined by Baykur [3]. Despite the strict conditions in the definition of this fibration, it is known that every closed oriented four-manifold admits a simplified broken Lefschetz fibration. For a simplified broken Lefschetz fibration, we can define a monodromy representation of this fibration as we do for a Lefschetz fibration. Thus we can define hyperelliptic simplified broken Lefschetz fibrations as a generalization of hyperelliptic Lefschetz fibrations. Hyperelliptic Lefschetz fibrations have been studied in many fields—for example, algebraic geometry and topology—and it has been shown that the total spaces of such fibrations satisfy strong conditions on the signature, the Euler characteristic, and so on (see e.g. [10]). Furthermore, we can obtain a signature formula of hyperelliptic simplified broken Lefschetz fibrations similar to that of hyperelliptic Lefschetz fibrations (see [13]). It is therefore natural to ask how far total spaces of hyperelliptic simplified broken Lefschetz fibrations are restricted as well as what conditions these spaces satisfy. The following result gives a partial answer.

Theorem 1.1. *Let $f: M \to S^2$ be a genus-g hyperelliptic simplified broken Lefschetz fibration. We assume that $g \geq 3$.*

(i) *Let s be the number of Lefschetz singularities of f whose vanishing cycles are separating. Then there exists an involution*

$$\omega: M \to M$$

such that the fixed point set of ω is the union of (possibly nonorientable) surfaces and s isolated points. Moreover, ω can be extended to an involution

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