Michigan Math. J. 62 (2013), 293–322

Boundary Complexes and Weight Filtrations

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1. Introduction

Let D be a divisor with simple normal crossings on an algebraic variety. The *dual complex* $\Delta(D)$ is a triangulated topological space, or Δ-complex, whose k-dimensional simplices correspond to the irreducible components of intersections of $k+1$ distinct components of D and where inclusions of faces correspond to inclusions of subvarieties; see Section 2 for further details. This paper studies the geometry and topology of dual complexes for boundary divisors of suitable compactifications as well as relations to Deligne's weight filtrations.

Let X be an algebraic variety of dimension n over the complex numbers. By theorems of Nagata [Na] and Hironaka [Hi], there is a compact variety $\bar{X}$ containing X as a dense open subvariety—and also a resolution

$$\varphi\colon X' \to \bar{X},$$

which is a proper birational morphism from a smooth variety that is an isomorphism over the smooth locus in X—such that the boundary

$$\partial X' = X' \setminus \varphi^{-1}(X)$$

and the union $\varphi^{-1}(X^{\text{sing}}) \cup \partial X'$ are divisors with simple normal crossings. We define the *boundary complex* of a resolution of a compactification, as just described, to be the dual complex $\Delta(\partial X')$ of the boundary divisor.

The intersections of irreducible components of boundary and exceptional divisors, along with the inclusions among them, encode a simplicial resolution of the pair $(\bar{X}, \bar{X} \setminus X)$ by smooth complete varieties; these data determine the weight filtration—and even the full mixed Hodge structure—on the cohomology of X. The combinatorial data in the boundary complex capture one piece of the weight filtration. Namely, there is a natural isomorphism from the reduced homology of the boundary complex to the $(2n)$th graded piece of the weight filtration on the cohomology of X:

$$\tilde{H}_{i-1}(\Delta(\partial X'); \mathbb{Q}) \cong \operatorname{Gr}^W_{2n} H^{2n-i}(X). \tag{1}$$

This isomorphism is well known to experts in mixed Hodge theory, and it was highlighted by Hacking for the case of X smooth [Hac, Thm. 3.1]. See Theorem 4.4 for the general case. The existence of such an isomorphism suggests that the topology of the boundary complex may be of particular interest.

Received December 23, 2011. Revision received February 20, 2012.