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Algebraic Montgomery–Yang Problem: The Nonrational Surface Case

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1. Introduction

A normal projective surface with the Betti numbers of the projective plane $\mathbb{CP}^2$ is called a *rational homology projective plane* or a $\mathbb{Q}$-*homology projective plane* or a $\mathbb{Q}$-*homology* $\mathbb{CP}^2$. When a normal projective surface S has rational singularities only, S is a $\mathbb{Q}$-homology projective plane if its second Betti number $b_2(S) = 1$. This can be seen easily by considering the Albanese fibration on a resolution of S.

It is known that a $\mathbb{Q}$-homology projective plane with quotient singularities (and no worse singularities) has at most five singular points (cf. [HK1, Cor. 3.4]). The authors have recently classified $\mathbb{Q}$-homology projective planes with five quotient singularities ([HK1]; also see [K2]).

In this paper we continue our study on the algebraic Montgomery–Yang problem, which was formulated by J. Kollár as follows.

Conjecture 1.1 [Kol2] (Algebraic Montgomery–Yang Problem). *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. Assume that $S^0 := S\setminus\mathrm{Sing}(S)$ is simply connected. Then S has at most three singular points.*

In [HK2] we confirm the conjecture when S has at least one noncyclic quotient singularity. Thus we may assume that S has cyclic singularities only. In this paper, we verify the conjecture when S is not rational.

Theorem 1.2. *Let S be a $\mathbb{Q}$-homology projective plane with cyclic singularities only. Assume that $H_1(S^0, \mathbb{Z}) = 0$. If S is not rational, then S has at most three singular points.*

Remark 1.3. The condition $H_1(S^0, \mathbb{Z}) = 0$ is weaker than the original condition $\pi(S^0) = \{1\}$, and there are infinitely many examples of $\mathbb{Q}$-homology projective planes with four quotient singularities—not all cyclic—such that $H_1(S^0, \mathbb{Z}) = 0$. Such $\mathbb{Q}$-homology projective planes are completely classified in [HK2]. It turns out that such a surface is a log del Pezzo surface with three cyclic singularities and

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