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Polyhedral Divisors and SL_2-Actions on Affine $\mathbb{T}$-Varieties

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Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0, let M be a lattice of rank n, let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice of M, and let $\mathbb{T}$ be the algebraic torus $\operatorname{Spec} \mathbf{k}[M]$ such that M is the character lattice of $\mathbb{T}$ and N is the 1-parameter subgroup lattice of $\mathbb{T}$.

A $\mathbb{T}$-*variety* X is a normal algebraic variety endowed with an effective regular action of $\mathbb{T}$. The *complexity* of a $\mathbb{T}$-action is the codimension of a general orbit; since the $\mathbb{T}$-action on X is effective, the complexity of X equals $\dim X - \operatorname{rank} M$. For an affine variety X, introducing a $\mathbb{T}$-action on X is the same as endowing $\mathbf{k}[X]$ with an M-grading. There are well-known combinatorial descriptions of $\mathbb{T}$-varieties. We refer the reader to [D1] and [Fu] for the case of toric varieties, to [K+, Chaps. 2 and 4] and [T2] for the complexity-1 case, and to [AH; AHS] for the general case. In this paper we use the approach in [AH].

Any affine toric variety is completely determined by a polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$. Similarly, the description of a normal affine $\mathbb{T}$-variety X due to Altmann and Hausen [AH] involves the data $(Y, \sigma, \mathfrak{D})$, where Y is a normal semiprojective variety, $\sigma \subseteq N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ is a polyhedral cone, and $\mathfrak{D}$ is a divisor on Y whose coefficients are polyhedra in $N_{\mathbb{Q}}$ with tail cone σ. The divisor $\mathfrak{D}$ is called a σ*-polyhedral divisor* on Y (see Section 1.1 for details).

Let X be a $\mathbb{T}$-variety endowed with a regular G-action, where G is any linear algebraic group. We say that the G-action on X is *compatible* if the image of G in $\mathrm{Aut}(X)$ is normalized but not centralized by $\mathbb{T}$. Furthermore, we say that the G-action is *of fiber type* if the general orbits are contained in the $\mathbb{T}$-orbit closures, and *of horizontal type* otherwise [FZ; L1].

Let now $\mathbb{G}_a = \mathbb{G}_a(\mathbf{k})$ be the additive group of $\mathbf{k}$. It is well known that a $\mathbb{G}_a$-action on an affine variety X is equivalent to a locally nilpotent derivation (LND) of $\mathbf{k}[X]$. A description of compatible $\mathbb{G}_a$-actions on an affine $\mathbb{T}$-variety—or, equivalently, of homogeneous LNDs on $\mathbf{k}[X]$—is available in the case where X is of complexity ≤ 1 [L1] or the $\mathbb{G}_a$-action is of fiber type [L2] in terms of a generalization of Demazure's [D1] roots of a fan (see Sections 1.3 and 1.4).

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