Michigan Math. J. 61 (2012), 715–729

Surfaces with Parallel Mean Curvature in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$

Dorel Fetcu & Harold Rosenberg

1. Introduction

In 1968, J. Simons discovered a fundamental formula for the Laplacian of the second fundamental form of a minimal submanifold in a Riemannian manifold. He then used this formula to characterize certain minimal submanifolds of a sphere and Euclidean space (see [17]). One year later, K. Nomizu and B. Smyth generalized Simons's equation for hypersurfaces of constant mean curvature (cmc hypersurfaces) in a space form (see [15]). This was extended, in Smyth's work [18], to the more general case of a submanifold with parallel mean curvature vector (pmc submanifold) in a space form. Over the years such equations, called Simons-type equations, turned out to be very useful, and a great number of authors have used them in the study of cmc and pmc submanifolds (see e.g. [3; 10; 16]).

Nowadays, the study of cmc surfaces in Euclidean space and, more generally, in space forms is a classical subject in the field of differential geometry; well-known papers by H. Hopf [13] and S.-S. Chern [9] are representative examples from the literature on this topic. When the codimension is greater than 1, a natural generalization of cmc surfaces are pmc surfaces. These have been intensively studied in the last four decades, and among the first papers devoted to this subject are those by D. Ferus [11], B.-Y. Chen and G. D. Ludden [8], D. A. Hoffman [12], and S.-T. Yau [20]. All results in these papers were obtained in the case when the ambient space is a space form.

The next natural step was taken by U. Abresch and H. Rosenberg, who studied in [1; 2] cmc surfaces and obtained Hopf-type results in product spaces $M^2(c) \times \mathbb{R}$, where $M^2(c)$ is a complete simply connected surface with constant curvature c, as well as in the homogeneous 3-manifolds $\mathrm{Nil}(3)$, $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$, and Berger spheres. Some of their results in [1] were extended to pmc surfaces in product spaces of $M^n(c) \times \mathbb{R}$, where $M^n(c)$ is an n-dimensional space form, by H. Alencar, M. do Carmo, and R. Tribuzy [4; 5].

In a recent paper, M. Batista [7] derived a Simons-type equation involving the traceless part of the second fundamental form of a cmc surface in $M^2(c) \times \mathbb{R}$ and found several applications.

Received April 14, 2011. Revision received May 16, 2012.

The first author was supported by a Postdoctoral Fellowship "Pós-Doutorado Sênior (PDS)", E-26/102.878/2010, offered by FAPERJ, Brazil.