Michigan Math. J. 61 (2012), 651–670

Parshin Residues via Coboundary Operators

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1. Introduction

Let X be a compact complex curve and let ω be a meromorphic 1-form on X. In an open neighborhood of each point $x \in X$ we can write

$$\omega = f(t)\,dt, \quad f(t) = \sum_{i>N} \lambda_i t^i,$$

where t is a local normalizing parameter at x. The coefficient λ_{-1} in the series does not depend on the choice of parameter t; it is called the *residue* of ω at x. The residue is nonzero only at the finitely many points $\Sigma \subset X$ where ω has a pole. The well-known residue formula states that the sum of residues of ω over all points of Σ is zero:

$$\sum_{x \in \Sigma} \operatorname{res}_x \omega = 0.$$

Indeed, the residue at $x \in \Sigma$ is equal to the integral of ω over any sufficiently small cycle enclosing x, divided by $2\pi i$. In the complement $X \setminus \Sigma$ the form ω is closed, and the sum of cycles is homologous to zero. Thus, the residue formula follows from the Stokes theorem.

Although this proof is topological, the residue itself can be defined purely algebraically. In fact, one can give an algebraic proof of the residue formula that works in a much more general situation, not only in the case of complex curves (see e.g. [S; T]).

In the late 1970s, Parshin introduced his notion of multidimensional residue for a rational n-form ω on an n-dimensional algebraic variety V_n. (Although [P] deals mostly with the 2-dimensional case, Beilinson [Bei] and Lomadze [L] generalized Parshin's ideas to the multidimensional case.) The main difference between the Parshin residue and the classical 1-dimensional residue is that, in higher dimensions, one computes the residue not at a point but instead at a complete flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$.

Parshin, Beilinson, and Lomadze proved the "reciprocity law" for multidimensional residues, which generalizes the classical residue formula and reads as follows.

> *Fix a partial flag of irreducible subvarieties* $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, *where* V_k *is omitted* $(0 < k < n)$. *Then*

Received April 25, 2011. Revision received October 21, 2011.