Michigan Math. J. 61 (2012), 543–553

# Betti Numbers of Smooth Schubert Varieties and the Remarkable Formula of Kostant, Macdonald, Shapiro, and Steinberg

Ersan Akyildiz & James B. Carrell

## 1. Introduction

Let $G$ be a semi-simple linear algebraic group over $\mathbb{C}$, $B$ a Borel subgroup of $G$, and $T \subset B$ a maximal torus. Let $\Phi$ denote the root system of the pair $(G, T)$ and $\Phi^+$ the set of positive roots determined by $B$. Let $\alpha_1, \dots, \alpha_\ell$ denote the basis of $\Phi$ associated to $\Phi^+$, and recall the *height* of $\alpha = \sum k_i \alpha_i \in \Phi$ is defined to be $\operatorname{ht}(\alpha) = \sum k_i$. Finally, let $W = N_G(T)/T$ be the Weyl group of $(G, T)$.

A remarkable formula—originally noticed by A. Shapiro and proved by Kostant [13] using the representation theory of the principal three-dimensional subgroup of $G$, by Macdonald [14] using the holomorphic Lefschetz formula, and by Steinberg [16] by verification—says that

$$\prod_{i=1}^{\ell}(1 + t^2 + \cdots + t^{2m_i}) = \prod_{\alpha \in \Phi^+} \frac{1 - t^{2\operatorname{ht}(\alpha)+2}}{1 - t^{2\operatorname{ht}(\alpha)}}, \tag{1}$$

where $m_1, \dots, m_\ell$ are the exponents of $G$. The identity (1) can also be formulated combinatorically. Suppose $h_i$ is the number of roots of height $i$ where $k$ is the height of the highest root. That is, $k + 1$ is the Coxeter number of $(G, T)$. Then $h_i \geq h_{i+1}$, so $(h_1, h_2, \dots, h_k)$ is a partition of $|\Phi^+|$. Then (1) is equivalent to saying that $(h_1, h_2, \dots, h_k)$ is conjugate to the partition determined by the exponents $m_j$ of $(G, T)$ (see Lemma 1).

A cohomological proof of (1), which we will generalize in this paper, goes as follows. First, by the well-known Borel picture of the cohomology algebra of $G/B$ as the coinvariant algebra of $W$, the Poincaré polynomial $P(G/B, t)$ of the flag variety $G/B$ has the expression

$$P(G/B, t) = \prod_{i=1}^{\ell} \frac{1 - t^{2d_i}}{1 - t^2}, \tag{2}$$

where $d_1, \dots, d_\ell$ are the degrees of the fundamental generators of the ring of $W$-invariant polynomials on the Lie algebra $\mathfrak{t}$ of $T$. By a different cohomological method, reviewed in Section 2 (cf. [1, Cor. 1]), one also obtains that

Received February 6, 2011. Revision received February 13, 2012.