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A Classification of Factorial Surfaces of Nongeneral Type

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1. Introduction

In this paper we deal exclusively with complex algebraic varieties.

An important invariant for a normal affine variety V is the logarithmic Kodaira dimension $\bar{\kappa}(V^{\circ})$ as defined by S. Iitaka, where V° is the smooth locus of V. A rich structure theory of smooth quasiprojective surfaces has been developed by S. Iitaka, Y. Kawamata, T. Fujita, M. Miyanishi, T. Sugie, S. Tsunoda, R. Kobayashi, and other Japanese mathematicians (for an excellent exposition, see [7]). We will use this theory and standard algebraic topology to give a geometric description of all 2-dimensional affine UFDs (or, factorial) V such that $\bar{\kappa}(V^{\circ})$ is at most 1. Many of the arguments in the proofs are by now standard (cf. [2, Sec. 3]).

This paper does not consider an algebraic description of the coordinate rings $\Gamma(V,\mathcal{O})$ of these unique factorization domains (UFDs). The multiplicative group of units in this ring will be denoted by $\Gamma(V,\mathcal{O})^*$.

We will prove the following four theorems.

Theorem 1. *Let V be a smooth, affine, factorial surface with $\Gamma(V,\mathcal{O})^* = \mathbb{C}^*$. Then we have the following assertions.*

- *If $\bar{\kappa}(V) = -\infty$, then $V \cong \mathbb{C}^2$.*
- *If $\bar{\kappa}(V) = 0$, then these surfaces are classified in* [3, Thm. 2] (*see Section 3*).
- *If $\bar{\kappa}(V) = 1$, then these surfaces are described in* [3] (*see Section 3*).

Remark. It is well known that any $\mathbb{Z}$-homology plane is factorial and has only trivial units.

Theorem 2. *Let V be an affine, factorial surface with at least one singular point and with $\Gamma(V,\mathcal{O})^* = \mathbb{C}^*$. Then we have the following assertions.*

- *If $\bar{\kappa}(V^{\circ}) = -\infty$ then $V \cong \mathbb{C}^2/\Gamma$, where Γ is the binary icosahedral group of order 120; hence V is the affine E_8-singularity $\{x^2 + y^3 + z^5 = 0\}$.*
- *If $\bar{\kappa}(V^{\circ}) = 0$, then V is obtained by one of the two constructions described in Section 4.*
- *If $\bar{\kappa}(V^{\circ}) = 1$, then V has a unique singular point with a good $\mathbb{C}^*$-action and $\Gamma(V,\mathcal{O})$ is a positively graded domain. These domains are all described by Mori in* [8] (*see Section 4*).

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