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Pairs of Additive Forms of Odd Degrees

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1. Introduction

A special case of a conjecture commonly attributed to Artin [1] states that if we consider a system of two additive homogeneous equations

$$\begin{aligned} a_1 x_1^k + a_2 x_2^k + \cdots + a_s x_s^k &= 0, \\ b_1 x_1^n + b_2 x_2^n + \cdots + b_s x_s^n &= 0 \end{aligned} \tag{1}$$

with all coefficients in $\mathbb{Q}$ and with $s \geq k^2 + n^2 + 1$, then this system should have a nontrivial solution in p-adic integers for each prime p. That is, the system should have a solution with at least one variable not equal to 0. Brauer [3] has demonstrated the existence of a finite bound on s in terms of k and n that guarantees nontrivial solutions, so the only question is whether the conjectured bound suffices. The purpose of this paper is to prove that it does when the degrees k and n are both odd.

In order to describe the previous work on this problem, we introduce a small amount of notation. For each prime p, we write $\Gamma_p^*(k,n)$ for the smallest value of s that guarantees the system (1) will have a nontrivial p-adic solution regardless of the values of the coefficients. Further, we define

$$\Gamma^*(k,n) = \max_{p \text{ prime}} \Gamma_p^*(k,n).$$

We will occasionally use implicitly the obvious facts that $\Gamma_p^*(k,n) = \Gamma_p^*(n,k)$ for each prime p and that $\Gamma^*(k,n) = \Gamma^*(n,k)$.

With this notation, the aforementioned result of Brauer shows that $\Gamma^*(k,n)$ exists for each pair of degrees; hence Artin's conjecture can be restated as claiming that one always has $\Gamma^*(k,n) \leq k^2 + n^2 + 1$. If we have only one homogeneous additive equation of degree k, then $\Gamma_p^*(k)$ and $\Gamma^*(k)$ are defined similarly. Davenport and Lewis [7] have shown that $\Gamma^*(k) \leq k^2 + 1$ for all k, with equality whenever $k + 1$ is prime, confirming another special case of Artin's conjecture.

Most previous work on the problem with two equations has dealt with the case where both forms have the same degree. If the degrees are equal and odd, then Davenport and Lewis [8] showed that the conjecture is true. If the degrees are equal and even, then Brüdern and Godinho [4] showed that if the degree cannot be written either as $p^\tau(p-1)$ with p prime and $\tau \geq 1$ or as $3 \cdot 2^\tau$ with $\tau \geq 1$,

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