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Monodromy Groups of Lagrangian Tori in $\mathbb{R}^4$

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1. Introduction

In this paper we work in the standard symplectic 4-space $\left(\mathbb{R}^4,\ \omega = \sum_{j=1}^{2} dx_j \wedge dy_j\right)$ unless otherwise mentioned. Let $L \overset{\iota}{\hookrightarrow} (\mathbb{R}^4, \omega)$ be an embedded Lagrangian torus with respect to the standard symplectic 2-form ω. The Lagrangian condition means that the pull-back 2-form $\iota^*\omega = 0 \in \Omega^2(L)$ vanishes on L. Gromov [7] proved that L is not *exact*—that is, the pull-back 1-form $\iota^*\lambda$ of a primitive λ of $\omega = d\lambda$ represents a nontrivial class in the cohomology group $H^1(L, \mathbb{R})$.

Let $\mathrm{Diff}_0^c(\mathbb{R}^4)$ denote the group of orientation-preserving diffeomorphisms with compact support on $\mathbb{R}^4$ that are isotopic to the identity map. We are interested in studying various types of self-isotopies of L. It is well known that to a smooth isotopy L_s, $s \in [0,1]$, between two embedded tori L_0, L_1 we may associate a family of maps $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ with $\phi_0 = \mathrm{id}$ such that $\phi_s(L) = L_s$. We will make no distinction between L_s and the associated maps ϕ_s from now on.

A path $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ with $0 \le s \le 1$ and $\phi_0 = \mathrm{id}$ associates to a fixed torus L a family of tori $L_s : \phi_s(L)$ in $\mathbb{R}^4$. The family of maps $\phi_t \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ is called a *smooth self-isotopy* of L if $\phi_1(L) = L$. Moreover, if all L_s are Lagrangian with respect to ω (ω-Lagrangian) then ϕ_s is called a *Lagrangian self-isotopy* of L. This is equivalent to saying that L is $\phi_s^*\omega$-Lagrangian. Suppose in addition that the cohomology class of $\iota^*\phi_s^*\lambda$ is independent of s; then ϕ_s is called a *Hamiltonian self-isotopy* of L. Equivalently, ϕ_s is Hamiltonian if it is generated by a Hamiltonian vector field. Each self-isotopy ϕ_s of L associates to an isomorphism

$$(\phi_1)_*\colon H_1(L, \mathbb{Z}) \to H_1(L, \mathbb{Z}),$$

which is called a *smooth* (resp., *Lagrangian, Hamiltonian*) monodromy of L if ϕ_t is smooth (resp., Lagrangian, Hamiltonian). The group of all smooth monodromies of L is called the *smooth monodromy group* (SMG) of L and is denoted by $\mathcal{S}(L)$. Likewise, $\mathcal{L}(L)$ and $\mathcal{H}(L)$ denote, respectively, the *Lagrangian monodromy group* (LMG) and the *Hamiltonian monodromy group* (HMG) of L. It is easy to see that $\mathcal{H}(L) \subset \mathcal{L}(L) \subset \mathcal{S}(L)$. Although here we focus only on Lagrangian 2-tori, the groups $\mathcal{H}(L)$, $\mathcal{L}(L)$, and $\mathcal{S}(L)$ are defined for any embedded Lagrangian submanifold L of any dimension.

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