Michigan Math. J. 61 (2012), 265–277

Discrepancies of Non-$\mathbb{Q}$-Gorenstein Varieties

Stefano Urbinati

1. Introduction

The aim of this paper is to investigate some surprising features of singularities of normal varieties in the non-$\mathbb{Q}$-Gorenstein case as defined by de Fernex and Hacon [dFH]. In that paper the authors focus on the difficulties of extending some invariants of singularities when the canonical divisor is not $\mathbb{Q}$-Cartier. Instead of the classical approach—-in which we modify the canonical divisor by adding a boundary, an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier—they introduce a notion of *pullback* of (Weil) $\mathbb{Q}$-divisors that agrees with the usual one for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. In this way, for any birational morphism of normal varieties $f : Y \to X$, they are able to define relative canonical divisors $K_{Y/X} = K_Y + f^*(-K_X)$ and $K^-_{Y/X} = K_Y - f^*(K_X)$. The two definitions coincide when K_X is $\mathbb{Q}$-Cartier; using $K_{Y/X}$ and $K^-_{Y/X}$, de Fernex and Hacon extended the definitions of canonical singularities, klt singularities, and multiplier ideal sheaves to this more general context.

In this setting, some of the properties characterizing the usual notions of singularity (see [KoMo, Sec. 2.3]) seem to fail owing to the asymptotic nature of the definitions of the canonical divisors.

We focus on three properties that for $\mathbb{Q}$-Gorenstein varieties are straightforward.

- The relative canonical divisor always has rational valuations (cf. [Ko2, Thm. 92]).
- A canonical variety is always kawamata log terminal (klt; cf. [KoMo, Def. 2.34]).
- The jumping numbers are a set of rational numbers that have no accumulation points (cf. [L2, Lemma 9.3.21]).

In this paper we investigate these properties for non-$\mathbb{Q}$-Gorenstein varieties. Section 2 is devoted to recalling the necessary definitions.

In Section 3, we show that if X is klt in the sense of [dFH] then the relative canonical divisor has rational valuations. We also give an example of a (non-klt) variety X with an irrational valuation and then use it to find an irrational jumping number (Theorem 3.6).

In Section 4 we give an example of a variety with canonical but not klt singularities (Theorem 4.1), and we prove that the finite generation of the canonical ring implies that the relative canonical model has canonical singularities (Proposition 4.4). Finally, in Section 5 we use one of the main results in [dFH]—namely, that every

Received October 4, 2010. Revision received December 8, 2010.