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Irregularity of the Bergman Projection on Worm Domains in $\mathbb{C}^n$

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2(\Omega)$ denote the Bergman space of square-integrable holomorphic functions on Ω. The Bergman projection on Ω is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$.

The Bergman projection is known to be regular in the sense that it maps W^s to W^s for all $s \geq 0$, where W^s denotes the Sobolev space of order s, on a large class of smooth bounded pseudoconvex domains (throughout this paper a domain is smooth if its boundary is a smooth manifold). Regularity is usually established through the $\bar\partial$-Neumann problem, the solution operator for the complex Laplacian $\square = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ on square-integrable $(0,1)$-forms. For more information on this matter we refer the reader to [BS3; S] and the references therein.

Irregularity of the Bergman projection is not understood nearly as well as regularity. The story of irregularity goes back to the discovery of the worm domains in $\mathbb{C}^2$ by Diederich and Fornæss [DF]. Worm domains were constructed to show that the closure of some smooth bounded pseudoconvex domains may not have Stein neighborhood bases (a compact set $K \subset \mathbb{C}^n$ is said to have a Stein neighborhood basis if for every open set U containing K there exists a pseudoconvex domain V such that $K \subset V \subset U$). Indeed, Diederich and Fornæss showed that the closure of a worm domain does not have a Stein neighborhood basis if the total winding is no less than π. It turned out that worm domains are also counterexamples for regularity of the Bergman projection. In 1991, Kiselman [Ki] showed that the Bergman projection does not satisfy Bell's condition R on nonsmooth worm domains (a domain Ω satisfies Bell's condition R if the Bergman projection maps $C^\infty(\bar\Omega)$ to $C^\infty(\bar\Omega)$). In 1992, Barrett [Ba] showed that the Bergman projection on a smooth worm domain does not map W^s into W^s if $s \geq \pi/(\text{total winding})$. On the other hand, Boas and Straube [BS2] showed that the Bergman projection maps W^k into W^k if $k \leq \pi/(2 \times \text{total winding})$ and k is a positive integer or $k = 1/2$. Finally, in 1996 Christ [Ch] showed that the Bergman projections on smooth worm domains with any positive winding do not satisfy Bell's condition R. More recently, Krantz and Peloso [KP1; KP2] studied the asymptotics for the Bergman

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