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A Relation between Height, Area, and Volume for Compact Constant Mean Curvature Surfaces in $\mathbb{M}^2 \times \mathbb{R}$

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1. Introduction

Let Σ be a compact CMC-H surface in $\mathbb{M}^2 \times \mathbb{R}$ with $\Gamma = \partial\Sigma \subset \mathbb{M}^2 \times \{0\}$, where $\mathbb{M}^2$ is a Hadamard surface with curvature $K_{\mathbb{M}^2} \leq -\kappa \leq 0$. Let Σ_1 be the connected component of the part of Σ above the plane $Q = \mathbb{M}^2 \times \{0\}$, and let h be the height of Σ_1 above Q. We will determine a volume V_1 bounded by Σ_1 and prove that

$$h \leq \frac{H|\Sigma_1|}{2\pi} - \frac{\kappa V_1}{4\pi};$$

here $|\Sigma_1|$ is the area of Σ_1. We also state conditions under which equality occurs.

We then let $\mathbb{M}^2 = \mathbb{H}^2$ be the hyperbolic plane of curvature -1, with $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ a compact CMC-H surface as just described. Finally, we give a condition that guarantees Σ lies in a half-space determined by Q.

We introduce some definitions and notation as follows. Let $\gamma \subset Q$ be a complete geodesic. We call $P = \gamma \times \mathbb{R}$ a *vertical plane* of $\mathbb{M}^2 \times \mathbb{R}$. Let $\beta(t)$ be a complete geodesic of Q, with $\beta(0)$ in the vertical plane P and $\beta'(0)$ orthogonal to P. Let $P_\beta(t)$ be the vertical plane of $\mathbb{M}^2 \times \mathbb{R}$ that passes through $\beta(t)$ and is orthogonal to β at $\beta(t)$. We call $P_\beta(t)$ the vertical plane *foliation* determined by P and β.

2. The Main Result

Let $\Sigma \subset \mathbb{M}^2 \times \mathbb{R}$ be a CMC-H surface as before and suppose that Σ meets Q transversally along $\Gamma = \partial\Sigma \subset Q$. We put $\Sigma^+ = \Sigma \cap (\mathbb{M}^2 \times \mathbb{R}_+)$ and $\Sigma^- = \Sigma \cap (\mathbb{M}^2 \times \mathbb{R}_-)$. There is a connected component of Σ^+ or Σ^- that contains Γ. We can assume, without loss of generality, that $\Gamma \subset \partial\Sigma^+$. We use Σ_1 to denote the connected component of Σ^+ that contains Γ.

Let $\hat{\Sigma}_1$ be the symmetry of Σ_1 through the plane Q. Then $\hat{\Sigma}_1 \cup \Sigma_1$ is a compact embedded surface with no boundary, and with corners along $\partial\Sigma_1$, that bounds a domain U in $\mathbb{M}^2 \times \mathbb{R}$. Let U_1 be the intersection of U with the half-space above Q. Thus U_1 is a bounded domain in $\mathbb{M}^2 \times \mathbb{R}$ whose boundary, ∂U_1, consists of the smooth connected surface Σ_1 and the union Ω of finitely smooth, compact and connected surfaces in Q. We define A^+ to be the area of Σ_1.

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