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Geometries of Lines and Conics on the Quintic del Pezzo 3-fold and Its Application to Varieties of Power Sums

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1. Introduction

1.1. Varieties of Power Sums

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19th century. This is called the *Waring problem* for a homogeneous form. We are interested in the study of the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. A precise definition of the claimed compactification is the following.

Definition 1.1.1. Let V be a $(v+1)$-dimensional vector space and let $F \in S^m \check{V}$ be a homogeneous form of degree m on V, where $\check{V}$ is the dual vector space of V. Let

$$\mathrm{VSP}(F,n)^o := \{(H_1,\dots,H_n) \mid H_1^m + \cdots + H_n^m = F\} \subset \mathrm{Hilb}^n(\mathbb{P}_*\check{V}).$$

The closed subset $\mathrm{VSP}(F,n) := \overline{\mathrm{VSP}(F,n)^o}$ is called the *varieties of power sums* of F.

Sometimes $\mathbb{P}_*\check{V}$ will be denoted by $\check{\mathbb{P}}^v$.

As far as we know, the first global descriptions of positive-dimensional VSPs were given by Mukai.

1.2. Mukai's Result

Let A_{22} be a smooth prime Fano 3-fold of genus 12—namely, a smooth projective 3-fold such that $-K_{A_{22}}$ is ample, the class of $-K_{A_{22}}$ generates $\mathrm{Pic}\, A_{22}$, and the genus $g(A_{22}) := (-K_{A_{22}})^3/2 + 1$ is equal to 12. The linear system $|-K_{A_{22}}|$ embeds A_{22} into $\mathbb{P}^{13}$.

Mukai discovered the following remarkable theorem [M1; M2].

Theorem 1.2.1. *Let $\{F_4 = 0\} \subset \mathbb{P}^2$ be a general plane quartic curve. Then*

(1) $\mathrm{VSP}(F_4, 6) \subset \mathrm{Hilb}^6\, \check{\mathbb{P}}^2$ *is an A_{22}; and, conversely,*
(2) *every general A_{22} is of this form.*

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