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A Multiplicative Formula for Structure Constants in the Cohomology of Flag Varieties

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1. Introduction

Let G be a connected, simply connected, semisimple complex algebraic group and let $P \subseteq Q$ be a pair of parabolic subgroups. Consider the induced sequence of flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q. \tag{1}$$

The goal of this paper is to give a simple multiplicative formula connecting the structure coefficients for the cohomology ring of the three flag varieties in (1) with respect to their Schubert bases. Let W be the Weyl group of G and let $W_P \subseteq W_Q \subseteq W$ denote the Weyl groups of P and Q, respectively. Let $W^P \subseteq W$ denote the set of minimal-length coset representatives in W/W_P. For any $w \in W^P$, let $\bar{X}_w \subseteq G/P$ denote the corresponding Schubert variety and let $[X_w] \in H^*(G/P) = H^*(G/P, \mathbb{Z})$ denote the Schubert class of $\bar{X}_w$. It is well known that the Schubert classes $\{[X_w]\}_{w \in W^P}$ form an additive basis for cohomology. Similarly, we have Schubert classes $[X_u] \in H^*(G/Q)$ for $u \in W^Q$ and $[X_v] \in H^*(Q/P)$ for $v \in W^P \cap W_Q$. The letters w, u, and v will be used to denote Schubert varieties in G/P, G/Q, and Q/P, respectively. In Lemma 2.1 we show that, for any $w \in W^P$, there is a unique decomposition $w = uv$ with $u \in W^Q$ and $v \in W^P \cap W_Q$. Fix $s \geq 2$ and, for any $w_1, \dots, w_s \in W^P$ such that $\sum_{k=1}^{s} \operatorname{codim} X_{w_k} = \dim G/P$, define the associated structure coefficient (or structure constant) to be the integer c_w for

$$[X_{w_1}] \cdots [X_{w_s}] = c_w[\mathrm{pt}] \in H^*(G/P).$$

The following theorem is the first result of this paper.

Theorem 1.1. *Let $w_1, \dots, w_s \in W^P$, and let $u_k \in W^Q$ and $v_k \in W^P \cap W_Q$ be defined by $w_k = u_k v_k$. Assume that*

$$\sum_{k=1}^{s} \operatorname{codim} X_{w_k} = \dim G/P \quad \textit{and} \quad \sum_{k=1}^{s} \operatorname{codim} X_{u_k} = \dim G/Q. \tag{2}$$

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ are defined by

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