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Extensions of Two Chow Stability Criteria to Positive Characteristics

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1. Introduction

We work over an algebraically closed field k of arbitrary characteristic. Let $X \subset \mathbb{P}_k^n$ be an effective cycle of dimension r and degree d in a projective space of dimension n. Analysis of the Chow (semi)stability of X is one of the basic problems in geometric invariant theory (GIT). Contrary to the case of asymptotic Chow (semi)stability, the precise classification of Chow (semi)stable cycles is quite a subtle problem and is known for only a few cases, even for projective hypersurfaces. For example, Shah [Sh] studied the case of plane sextics and Laza [La] studied the case of cubic 4-folds—both in relation to period maps.

On the other hand, there are two sufficient conditions for Chow (semi)stability in terms of the singularity of X or that of the Chow divisor $Z(X) \subset \mathbb{G} = \mathrm{Grass}_k(n-r, n+1)$, which deal with general situations. Both have been proved in characteristic 0, and the purpose of this paper is to extend them to arbitrary characteristics. Namely, we prove the following two theorems.

Theorem 1.1 (= Theorem 3.1). *If $d \geq 3$, then any nonsingular projective hypersurface of degree d is Chow stable.*

Theorem 1.2 (= Theorem 4.1). *Let X be an effective cycle of dimension r and degree d in $\mathbb{P}_k^n$. Let $(\mathbb{G}, Z(X))$ be the log pair defined by the Chow divisor $Z(X)$ of X. If $\mathrm{lct}(\mathbb{G}, Z(X))$ is greater than (respectively, is greater than or equal to) $\frac{n+1}{d}$, then X is Chow stable (respectively, Chow semistable).*

In the statement of Theorem 1.2, $\mathrm{lct}(\mathbb{G}, Z(X))$ denotes the log canonical threshold of $(\mathbb{G}, Z(X))$, which measures how good the singularity of $Z(X)$ is (see Section 2.2 for details). The characteristic-0 case of Theorem 1.1 is due to Mumford [GIT, Chap. 4, Sec. 2], and that of Theorem 1.2 is due to Lee [Le].

The original proof of Theorem 1.1 works only when the characteristic of the base field does not divide d (see Section 3). To prove the general case, we depend on the corresponding result in characteristic 0.

We sketch the proof of Theorem 1.1 in positive characteristic. First we take a suitable lift of the equation of a given hypersurface over the ring of Witt vectors. This defines a family of projective hypersurfaces over the ring. We are assuming that the closed fiber is nonsingular; hence the geometric generic fiber is also

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