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Hodge Polynomials of Singular Hypersurfaces

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1. Introduction and Statement of Results

Let X be an n-dimensional compact complex algebraic manifold and $\mathcal{L}$ a line bundle on X. Let $\mathbf{L} \subset \mathbb{P}(H^0(X, \mathcal{L}))$ be a line in the projectivization of the space of sections of $\mathcal{L}$ (i.e., a pencil of hypersurfaces in X). Assume that the generic element L_t in $\mathbf{L}$ is nonsingular and that L_0 is a singular element of $\mathbf{L}$. The purpose of this paper is to relate the Hodge polynomials of the singular and (respectively) generic member of the pencil—in other words, to understand the difference $\chi_y(L_0) - \chi_y(L_t)$ in terms of invariants of the singularities of L_0. A special case of this situation was considered by Parusiński and Pragacz in [PPr1], who studied the topological Euler characteristic of pencils for which the generic element L_t of the pencil $\mathbf{L}$ is transversal to the strata of a Whitney stratification of L_0. This led the authors of [PPr1] to a calculation of Parusiński's generalized Milnor number (see [P]) of a singular hypersurface and also to a characteristic class version of this formula in [PPr2] for the Chern–Schwartz–MacPherson classes (see [Mac]). In a different vein, the Hodge theory of 1-parameter degenerations was considered in [CLMSh2] (cf. [Di1] for the case when L_0 has only isolated singularities) by using Hodge-theoretical aspects of the nearby and vanishing cycles associated to the degenerating family of hypersurfaces and extending similar Euler characteristic calculations presented earlier in [Di2]. This paper adds an extra layer of complexity by addressing the Hodge-theoretic situation in the context of a pencil of hypersurfaces with nonempty base locus.

Let us first define the invariants to be investigated. A *functorial χ_y-genus* is defined by the ring homomorphism

$$\chi_y\colon K_0(\mathrm{MHS}) \to \mathbb{Z}[y, y^{-1}]; \quad [V] \mapsto \sum_p \dim_{\mathbb{C}} \mathrm{Gr}_F^p(V \otimes_{\mathbb{Q}} \mathbb{C}) \cdot (-y)^p, \tag{1.1}$$

where $K_0(\mathrm{MHS})$ is the Grothendieck ring (with respect to the tensor product) of the abelian category MHS of rational mixed Hodge structures [De1; De2] and $\mathrm{Gr}_F^p(V \otimes_{\mathbb{Q}} \mathbb{C}) := F^p/F^{p+1}$ ($p \in \mathbb{Z}$) denotes the pth graded part of the (decreasing) Hodge filtration $F^\bullet$ corresponding to the mixed Hodge structure $V \in \mathrm{MHS}$. For $K^\bullet \in D^b(\mathrm{MHS})$ a bounded complex of rational mixed Hodge structures, we set $[K^\bullet] := \sum_i (-1)^i [K^i] \in K_0(\mathrm{MHS})$ and define

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