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Normal Forms, Hermitian Operators, and CR Maps of Spheres and Hyperquadrics

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1. Introduction

The purpose of this paper is to use normal forms of Hermitian operators in the study of CR maps. We strengthen a useful link between linear algebra and several complex variables and apply the techniques discussed to the theory of rational CR maps of spheres and hyperquadrics. After discussing the classification of such CR maps in terms of their Hermitian forms, we turn to the classification of Hermitian forms arising from degree-2 rational CR maps of spheres and hyperquadrics. For degree-2 maps, the classification is the same as that of a pair of Hermitian forms up to simultaneous $*$-congruence (matrices A and B are $*$-congruent if there exists a nonsingular matrix X such that $X^*AX = B$, where X^* is the conjugate transpose), which is a classical problem in linear algebra whose solution dates to the 1930s (see the survey [18]).

We will apply the theory developed in this paper to two problems. First, extending a result of Faran [12], in Theorem 1.2 we finish the classification of all CR maps of hyperquadrics in dimensions 2 and 3. Second, Ji and Zhang [17] classified degree-2 rational CR maps of spheres from source dimension 2. In Theorem 1.5 we extend this result to arbitrary source dimension and give an elegant version of the theorem by proving that all degree-2 rational CR maps of spheres in any dimension are spherically equivalent to a monomial map. We also study the real-algebraic version of the CR maps of hyperquadrics problem in dimensions 2 and 3, which arises in the case of diagonal Hermitian forms.

In CR geometry we often think of a real-valued polynomial $p(z,\bar{z})$ on complex space as the composition of the Veronese map $\mathcal{Z}$ with a Hermitian form B. That is, $p(z,\bar{z}) = \langle B\mathcal{Z}, \mathcal{Z}\rangle$. See Section 2 for more on this setup. Writing B as a sum of rank-1 matrices, we find that p can also be viewed as the composition of a holomorphic map composed with a diagonal Hermitian form. When we divide the form $\langle B\mathcal{Z}, \mathcal{Z}\rangle$ by the defining equation of the source hyperquadric, the result is a pair of Hermitian forms. When this pair is put into canonical form, we obtain a canonical form of the map up to a natural equivalence relation. A crucial point is that, for degree-2 maps, the two forms are linear in z.

We thus make a connection between real polynomials and holomorphic maps to hyperquadrics. A hyperquadric is the zero set of a diagonal Hermitian form and

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