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Cubic Relations between Frequencies of Digits and Hausdorff Dimension

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1. Introduction

We consider the notion of frequency of digits, which is defined as follows. Given an integer $m > 1$, for each $x \in [0,1]$ we denote by $0.x_1x_2\cdots$ a base-m representation of x (the representation is unique except for countably many points, and thus the nonuniqueness does not affect the study of Hausdorff dimension, since countable sets have zero Hausdorff dimension). For each $k \in \{0,\dots,m-1\}$, $x \in [0,1]$, and $n \in \mathbb{N}$ we set

$$\tau_k(x,n) = \operatorname{card}\{i \in \{1,\dots,n\} : x_i = k\}$$

and

$$\tau_k(x) = \lim_{n\to\infty} \frac{\tau_k(x,n)}{n}$$

whenever the limit exists. The number $\tau_k(x)$ is called the *frequency* of the number k in the base-m representation of x. Now we consider the set

$$F_m(\alpha_0,\dots,\alpha_{m-1}) = \{x \in [0,1] : \tau_k(x) = \alpha_k \text{ for } k = 0,\dots,m-1\},$$

where $\alpha_k \in [0,1]$ for each k and $\sum_{k=0}^{m-1} \alpha_k = 1$. Eggleston showed in [11] that this set has Hausdorff dimension

$$\dim_H F_m(\alpha_0,\dots,\alpha_{m-1}) = -\frac{1}{\log m}\sum_{k=0}^{m-1} \alpha_k \log \alpha_k$$

(with the convention that $0\log 0 = 0$). A related result was obtained earlier by Besicovitch in [6] when $m = 2$. The work of Eggleston was further generalized by Billingsley with a more unified approach (see [7] for details and references). See also [3] for more recent developments and further references.

In this paper we consider sets defined in terms of *nonlinear* relations between the frequencies. Namely, for each ε and δ we consider the function

$$f_{\varepsilon,\delta}(x) = x + \varepsilon x^2 + \delta x^3$$

and the set

$$F_{\varepsilon,\delta} = \{x \in [0,1] : \tau_1(x) = f_{\varepsilon,\delta}(\tau_0(x))\}. \tag{1}$$

It follows from a general procedure described in [3] that

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