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Polarized Complexity-1 T-Varieties

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Introduction

It is well known that there is a correspondence between polarized toric varieties and lattice polytopes. The main result of this paper is to generalize this to the setting of normal varieties with effective complexity-1 torus action—that is, complexity-1 T-varieties. In order to do so, we introduce so-called *divisorial polytopes.* In short, a divisorial polytope on a smooth projective curve Y in a lattice M is a piecewise affine concave function

$$\Psi = \sum_{P \in Y} \Psi_P \cdot P : \Box \to \operatorname{Div}_{\mathbb{Q}} Y$$

from some polytope in $M_{\mathbb{Q}}$ to the group of $\mathbb{Q}$-divisors on Y such that:

(i) $\deg \Psi(u) > 0$ for u from the interior of $\Box$;
(ii) $\deg \Psi(u) > 0$ or $\Psi(u) \sim 0$ for u a vertex of $\Box$; and
(iii) the graph of Ψ_P has integral vertices for every $P \in Y$.

We then show that, similarly to the toric case, there is a correspondence between polarized complexity-1 T-varieties and divisorial polytopes. We also describe how the smoothness, degree, and Hilbert polynomial of a polarized T-variety can be determined from the corresponding divisorial polytope.

There are two other logical approaches to describing a polarized complexity-1 T-variety. Indeed, T-invariant Cartier divisors on complexity-1 T-varieties were described in terms of *divisorial fans* and support functions in [PS], which also included a characterization of ampleness. On the other hand, a sufficiently high multiple of some polarizing line bundle gives a map to projective space such that the corresponding affine cone is a complexity-1 T-variety describable by a *polyhedral divisor* $\mathcal{D}$. We compare these two approaches with our divisorial polytopes and show how to pass from one description to another.

We also present two other results. First, we show how the complicated combinatorial data of a divisorial fan used to describe a general T-variety can be simplified to a so-called *marked fansy divisor* for complete complexity-1 T-varieties. Second, we address the problem of finding minimal generators for the multigraded $\mathbb{C}$-algebra corresponding to a polyhedral divisor $\mathcal{D}$ on a curve. This then gives us a method to determine whether projective embeddings of complexity-1 T-varieties are, in fact, projectively normal.

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