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The Generalized Oka–Grauert Principle for 1-Convex Manifolds

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1. Introduction and Main Theorem

Let Y be a complex manifold satisfying the convex approximation property (CAP) and let X be an arbitrary Stein manifold. Then the Oka–Grauert (or homotopic) principle holds for mappings $X \to Y$ ([F2]; the name *Oka manifold* has recently been suggested for such manifolds Y). This means that each homotopy class of mappings $X \to Y$ admits a holomorphic representative.

Manifolds satisfying CAP are in some sense "large". As an example of a manifold failing to satisfy CAP, consider the annulus $Y = \{z \in \mathbb{C}, 1/2 < |z| < 2\}$. Let $X = \{z, 1/3 < |z| < 3\}$. There are plenty of continuous mappings from X to Y but no nontrivial holomorphic ones. The reason is that Y is "too small" for X. If we are free to change the holomorphic structure on X, then we can find for every continuous mapping $f_0 \colon X \to Y$ another Stein structure J_1 on X that is homotopic to the initial one as well as a holomorphic mapping $f_1 \colon (X, J_1) \to Y$ in the same homotopy class as f_0. In general, the change of structure depends on both Y and f_0. In the simple example just given, the manifold X is homotopically equivalent to the unit circle $S^1 \subset X$ and we change the homotopic structure of X simply by squeezing it diffeotopically into a small neighborhood of the unit circle. For a general Stein manifold X we can proceed analogously, replacing S^1 by a suitably fattened CW complex embedded in X and homotopically equivalent to X to obtain the following.

Generalized Oka–Grauert Principle. Every continuous mapping $X \to Y$ from a Stein manifold X to a complex manifold Y is homotopic to a holomorphic one provided that either Y satisfies CAP or we are free to change the complex structure on X [FS1]. In addition, we can also require that the structure is fixed on a neighborhood of an analytic set $X_0 \subset X$ if the initial mapping is holomorphic on a neighborhood of X_0 [FS2].

It has recently been shown that, if X is 1-convex and Y satisfies CAP, then the following version of the Oka–Grauert principle holds.

Relative Oka–Grauert Principle for Mappings. Every continuous mapping $X \to Y$ from a 1-convex manifold X to a complex manifold Y that satisfies

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