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Fano Surfaces with 12 or 30 Elliptic Curves

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Introduction

A Fano surface is a surface of general type that parameterizes the lines of a smooth cubic threefold. First studied by Fano and then by many others—including Bombieri and Swinnerton-Dyer [4], Gherardelli [7], Tyurin [14; 15], Clemens and Griffiths [5], and Collino [6]—these surfaces carry many remarkable properties. In our previous paper [12], we classified Fano surfaces according to the configurations of their elliptic curves. The aim of the present paper is to give various applications of this study when the Fano surface contains 12 or 30 elliptic curves.

The main result of the first part of this paper is as follows.

Proposition 1.

(i) *The Picard number* ρ_S *of a Fano surface* S *satisfies* $1 \leq \rho_S \leq 25$ *and is* 1 *for* S *generic.*

(ii) *A Fano surface that contains* 12 *elliptic curves is a triple ramified cover of the blow-up of* 9 *points of an abelian surface.*

(iii) *The Néron–Severi group of such a surface has rank* 12, 13 *or* $25 = h^{1,1}(S)$.

(iv) *For* S *generic among Fano surfaces with* 12 *elliptic curves, the Néron–Severi group has rank* 12 *and is rationally generated by its* 12 *elliptic curves.*

(v) *An infinite number of Fano surfaces with* 12 *elliptic curves have maximal Picard number* $25 = h^{1,1}(S)$.

Recall that among the K3 surfaces, the Kummer surfaces are recognized as those K3 having 16 disjoint (-2)-curves. They are the double cover of the blow-up over the 2-torsion points of an abelian surface (see [11]). Our theorem is the analogue for Fano surfaces that contains 12 elliptic curves among Fano surfaces.

In the second part, we study the Fano surface S of the Fermat cubic threefold $F \hookrightarrow \mathbb{P}^4$:

$$F = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}.$$

Let μ_3 be the group of third roots of unity and let $\alpha \in \mu_3$ be a primitive root. For s a point of S, we denote by L_s the line on F corresponding to the point s and we denote by C_s the incidence divisor that parameterizes the lines in F that cut the line L_s.

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