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Plane Sextics with a Type-$\mathbf{E}_6$ Singular Point

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1. Introduction

1.1. The Subject

This paper concludes the series [11; 12], where we give a complete deformation classification and compute the fundamental groups of maximizing irreducible plane sextics with an $\mathbf{E}$-type singular point. (With the common abuse of the language, by the *fundamental group* of a curve $B \subset \mathbb{P}^2$ we mean the group $\pi_1(\mathbb{P}^2 \setminus B)$ of its complement.) Here, we consider sextics $B \subset \mathbb{P}^2$ satisfying the following conditions:

($*$) B has simple (i.e., $\mathbf{A}$–$\mathbf{D}$–$\mathbf{E}$) singularities only,
B has a distinguished singular point P of type $\mathbf{E}_6$, and
B has no singular points of type $\mathbf{E}_7$ or $\mathbf{E}_8$.

(Singular points of type $\mathbf{E}_7$ and $\mathbf{E}_8$ are excluded in order to reduce the lists. Sextics with a type-$\mathbf{E}_8$ point are considered in [12], and irreducible sextics with a type-$\mathbf{E}_7$ point are considered in [11]. Reducible sextics with a type-$\mathbf{E}_7$ point, as well as the more involved case of a distinguished $\mathbf{D}$-type point, may appear elsewhere.)

Recall that a plane sextic B with simple singularities only is called *maximizing* if the total Milnor number $\mu(B)$ assumes the maximal possible value 19. It is well known that maximizing sextics are defined over algebraic number fields (as they are related to singular $K3$-surfaces). Furthermore, such sextics are rigid: two maximizing sextics are equisingular deformation equivalent if and only if they are related by a projective transformation.

Another important class is formed by the so-called sextics of *torus type*—in other words, those whose equation can be represented in the form $f_2^3 + f_3^2 = 0$, where f_2 and f_3 are certain homogeneous polynomials of degree 2 and 3, respectively. (This property turns out to be equisingular deformation invariant.) Each sextic B of torus type can be perturbed to Zariski's famous six-cuspidal sextic [21], which is obtained when f_2 and f_3 as just described are sufficiently generic. Hence, the group $\pi_1(\mathbb{P}^2 \setminus B)$ factors to the *reduced braid group* $\bar{\mathbb{B}}_3 := \mathbb{B}_3/(\sigma_1\sigma_2)^3$; in particular, it is never finite. (The existence of two distinct families of irreducible six-cuspidal sextics, those of and not of torus type, was first stated by Del Pezzo and then proved by Segre; see e.g. [19, p. 407]. Zariski [21] later showed that the two families differ by the fundamental groups.)

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