Michigan Math. J. 60 (2011), 215–239

Hyperbolizing Hyperspaces

Zair Ibragimov

Dedicated to the bright memory of Juha Heinonen

1. Introduction

The aims of this paper are to establish connections between a metric space X and the large-scale geometry (in the sense of Gromov) of the hyperspace $\mathcal{H}(X)$ of its nondegenerate closed bounded subsets and to study mappings on X in terms of the induced mappings on $\mathcal{H}(X)$. The metric space X can be identified with the boundary of $\mathcal{H}(X)$ when the latter is equipped with the Hausdorff metric, but stronger relationships between X and $\mathcal{H}(X)$ are obtained when the hyperspace $\mathcal{H}(X)$ is hyperbolized and the space X is identified with its boundary at infinity: a priori weak conditions on $\mathcal{H}(X)$ are strengthened at the boundary at infinity. The basic tool for studying such relationships is Gromov's theory of negatively curved spaces [22]. These spaces, known as Gromov hyperbolic spaces, are important in many areas of analysis and geometry, including geometric function theory, geometric group theory, and analysis on metric spaces. Another tool comes from the uniformization theory of Bonk, Heinonen, and Koskela [7]. It provides tools for hyperbolizing $\mathcal{H}(X)$ in such a way that the resulting space is complete, proper, geodesic, hyperbolic, and such that the boundary at infinity is identified with X.

One of the advantages of using the hyperspace $\mathcal{H}(X)$ is the associated extension operator; any injective map $f\colon X \to Y$ that maps closed bounded sets to closed bounded sets has a canonical extension to a map $\hat{f}\colon \mathcal{H}(X) \to \mathcal{H}(Y)$ defined by $\hat{f}(A) = f(A)$. When the hyperspaces are endowed with the Hausdorff metric, the map $\hat{f}$ is not generally continuous even if f is. Therefore, it is more natural to study $\hat{f}$ within the context of Gromov hyperbolic spaces—once the hyperspaces are hyperbolized. One of the useful features of the extension $f \mapsto \hat{f}$ is its compatibility under composition. That is, $\widehat{f \circ g} = \hat{f} \circ \hat{g}$. This paves the way to a study of groups acting on X by extending them to groups acting on $\mathcal{H}(X)$ and studying the latter within the theory of Gromov hyperbolic spaces.

Let us consider Euclidean space $\mathbb{R}^n$. The one-point compactification $\overline{\mathbb{R}^n}$ is equipped with the chordal metric, which is Möbius equivalent to the Euclidean metric when restricted to $\mathbb{R}^n$. The space $\overline{\mathbb{R}^n}$ can be identified with the ideal boundary of the hyperbolic space $\mathbb{H}^{n+1}$ and, as such, it inherits a family of visual metrics from $\mathbb{H}^{n+1}$. In fact, the chordal metric is one such visual metric. The Möbius transformations of $\overline{\mathbb{R}^n}$ can be extended to isometries of $\mathbb{H}^{n+1}$ by the Poincaré extension

Received June 30, 2009. Revision received July 9, 2010.